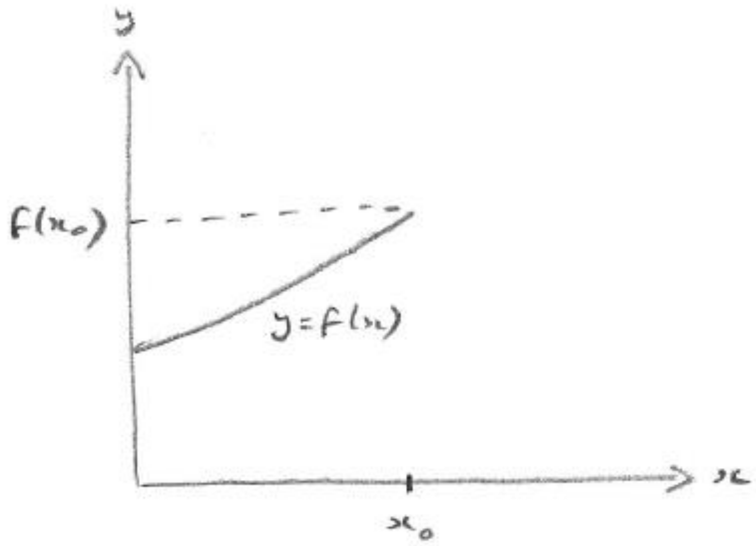
MAT Problems E - Inequalities (Sol'ns) (6 pages; 19/9/17)

(1) If $f(x) \leq f(x_0)$ and $f(x) \leq a + \frac{1}{2}f(x_0)$, show that $f(x) \leq 2a$

Solution

[At first sight, this doesn't look promising, as the inequalities in seem to be in unfavourable directions:

$f(x) \leq a + \frac{1}{2}f(x_0) \Rightarrow f(x_0) \geq 2f(x) - 2a$, but this can't be usefully combined with $f(x_0) \geq f(x)$



However, if we consider a simple example of a graph of $f(x)$, with an upper limit of $f(x_0)$ [see diagram], and note that $f(x)$ can't be above $a + \frac{1}{2}f(x_0)$, then we see that this doesn't work if $f(x_0)$ is very large relative to a, but that it can do if $f(x_0)$ is small enough relative to a

(in general, a useful device is to consider extreme situations)

So we need to be looking for an upper limit for $f(x_0)$.

As $f(x) \leq a + \frac{1}{2}f(x_0)$, $f(x_0) \leq a + \frac{1}{2}f(x_0)$,

so that $\frac{1}{2}f(x_0) \leq a$ and $f(x_0) \leq 2a$

Then $f(x) \leq f(x_0) \leq 2a$, as required.

(2) Given that $f(x)$ has a maximum on the interval $0 \leq x \leq \frac{1}{2}$ at $x = x_0$, show that $\int_0^x f(t)dt \leq \frac{1}{2}f(x_0)$ whenever $0 \leq x \leq \frac{1}{2}$

Solution

Consider the area under the graph of $f(t)$, between 0 & x.

Assume for the moment that the graph lies above the t-axis.

The maximum height of the function is $f(x_0)$, and the area under the graph is no greater than the rectangle with base x and height $f(x_0)$.

As $x \leq \frac{1}{2}$, the rectangle has area $\leq \frac{1}{2}f(x_0)$.

As the integral would have a smaller value if part of the graph were to lie below the t-axis,

$\int_0^x f(t)dt \leq \frac{1}{2}f(x_0)$ whenever $0 \leq x \leq \frac{1}{2}$

(3) Assuming that $sin^2\theta + cos^2\theta = 1$, but without using any compound angle results, show that $sin\theta cos\theta \leq \frac{1}{2}$

Solution

$(sin\theta - cos\theta)^2 \geq 0 \Rightarrow sin^2\theta + cos^2\theta - 2sin\theta cos\theta \geq 0$

$\Rightarrow 1 \geq 2sin\theta cos\theta \Rightarrow sin\theta cos\theta \leq \frac{1}{2}$

(4) Which is larger: $\frac{\sqrt{7}}{2}$ or $\frac{1+\sqrt{6}}{3}$ (without using a calculator)?

Solution

Considering the difference of squares:

$\frac{7}{4} - \frac{(1+2\sqrt{6}+6)}{9} = \frac{63-28-8\sqrt{6}}{36} > \frac{35-8(3)}{36} > 0$; so $\frac{\sqrt{7}}{2}$ is larger

[Another approach is to investigate $\frac{\left(\frac{7}{4}\right)}{\left(\frac{7+2\sqrt{6}}{9}\right)} = \frac{63(7-2\sqrt{6})}{4(49-24)} =$ $\frac{63(7-2\sqrt{6})}{100}$, but it isn't as easy to show that this expression is greater than 1]

(5) How would you solve the inequality: $\frac{1}{x} < x$?

Solution

Method 1: Multiply both sides by x^2

Method 2: Treat the cases $x < 0$ and $x > 0$ separately

Method 3: Rearrange as $\frac{1}{x} - x < 0$

Method 4: Sketch $y = \frac{1}{x}$ and $y = x$, and consider points of intersection

(6) Is $\frac{6}{7} < \frac{2}{\sqrt{5}}$?

Solution

$\frac{6}{7} < \frac{2}{\sqrt{5}} \Leftrightarrow \frac{36}{49} < \frac{4}{5}$

$49 \times 0.8 = \frac{1}{10}(320 + 72) = 39.2 > 36$

So $\frac{36}{49} < \frac{39.2}{49} = 0.8 = \frac{4}{5}$

Answer is Yes.

(7) Is $log_2 3 > \frac{3}{2}$?

Solution

$log_2 3 > \frac{3}{2} \Leftrightarrow 3 > 2^{\frac{3}{2}}$ (as $y = 2^x$ is an increasing function)

$\Leftrightarrow 3^2 > 2^3$

So answer is Yes.

(8) Are the following true or false?

(i) $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$

(ii) $a < b \Rightarrow a^2 < b^2$

(iii) $a < b \;\&\; c < d \Rightarrow a + c < b + d$

(iv) $a < b \;\&\; c < d \Rightarrow a - c < b - d$

Solution

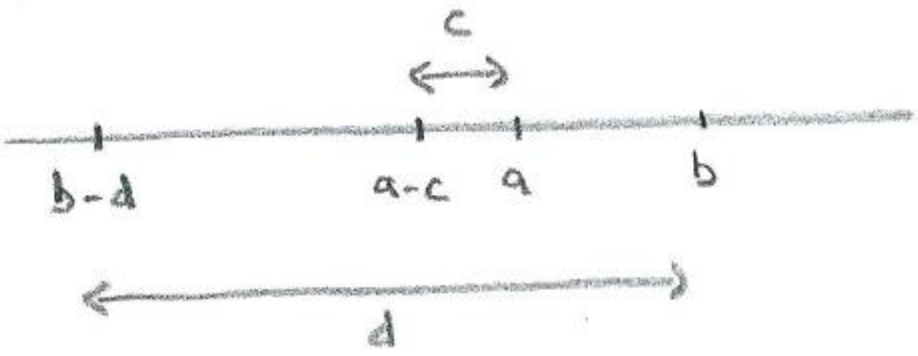
(i) Not true if $a < 0 \;\&\; b > 0$ (consider the graph of $y = 1/x$)

(ii) Not true if $a < 0 \;\&\; b < 0$ or

if $a < 0, b > 0 \;\&\; |b| < |a|$(consider the graph of $y = x^2$)

(iii) True: $a < b \Rightarrow a + c < b + c < b + d$

(iv) False: For example, $8 < 9$ and $2 < 4$, but it is not true that $8 - 2 < 9 - 4$; see diagram



(9) Prove or provide a counter-example for the conjecture

$x > a \;\&\; y > b \Rightarrow xy > ab \quad (a, b\ real)$ in each of the following cases:

(i) $a > 0, b > 0$ (ii) $a < 0, b < 0$ (iii) $a > 0, b < 0$

Solution

(i) $x > a \Rightarrow xy > ay$ [as $y > 0$] $> ab$ [since $y > b \Rightarrow ay > ab$]

so true

[or refer to graph of $y = ab$]

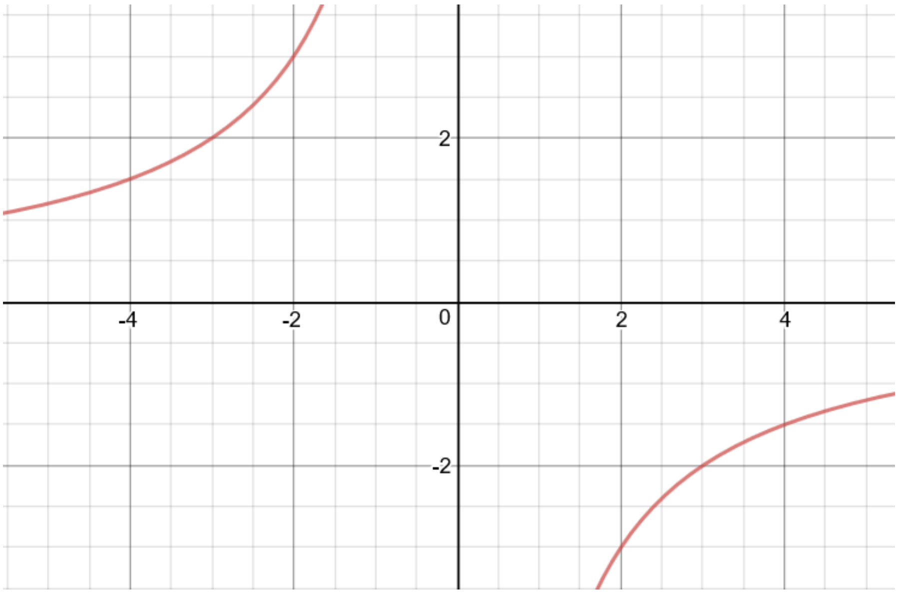
(b) $a < 0, b < 0$

counter-example: $x = 0$

(c) $a > 0, b < 0$

consider graph of $xy = ab$ when $a = 3, b = -2$ (see below)

(counter-example: $x = 4 + \delta, y = -2 + \delta$)


2
-2
-4
-2
0
2
4