MAT Problems D - Algebra (Sol'ns) (4 pages; 19/9/17)

(1) Solve the equation $x - \sqrt{x} = 6$

Solution

Let $f(x) = x - \sqrt{x} - 6$

$f(x) = 0 \Rightarrow x - 6 = \sqrt{x}$

$\Rightarrow (x-6)^2 = x$, but this may include spurious solutions

[of $x - 6 = -\sqrt{x}$]

$\Rightarrow x^2 - 13x + 36 = 0$

$\Rightarrow (x-9)(x-4) = 0$

$\Rightarrow x = 9\ or\ x = 4$

$f(9) = 0$ & $f(4) = -4$

Thus the only solution is $x = 9$

[Let $g(x) = x + \sqrt{x} - 6 = 0$

Then $g(x) = 0 \Rightarrow (x-6)^2 = x$ as well

$g(9) \neq 0, and\ g(4) = 0$]

Alternatively: Let $y = \sqrt{x}$, so that

$x - \sqrt{x} - 6 = 0 \Rightarrow y^2 - y - 6 = 0$

$\Rightarrow (y+2)(y-3) = 0$

$\Rightarrow y = -2$ $\left(\text{reject as } \sqrt{\text{x}} \text{ must be} \geq 0\right)$ $or\ y = 3$

(2) Given that $\frac{bc-a}{1-c} = 7$ & $\frac{b^2c-a^2}{1-c} = 51$, show that $\frac{a+7}{a^2+51} = \frac{b+7}{b^2+51}$

Solution

$\frac{bc-a}{1-c} = 7 \Rightarrow bc - a = 7 - 7c \Rightarrow c(b+7) = 7 + a$

$\Rightarrow c = \frac{a+7}{b+7}$

and replacing a, b & 7 with a^2, b^2 & 51 gives $c = \frac{a^2+51}{b^2+51}$

so that $\frac{a+7}{b+7} = \frac{a^2+51}{b^2+51}$ and hence $\frac{a+7}{a^2+51} = \frac{b+7}{b^2+51}$ (since $a^2 + 51$ & $b^2 + 51$ are both non-zero)

(3) (i) Find an expansion for $(a+b+c)^3$, and give a justification for the coefficients.

(ii) Extend this to $(a+b+c)^4$

Solution

(i) By an ordinary expansion:

$(a+b+c)^3 = ([a+b]+c)^3$

$= (a+b)^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3$

$= (a^3 + 3a^2b + 3ab^2 + b^3) + (3a^2c + 3b^2c + 6abc)$

$+(3ac^2 + 3bc^2) + c^3$

$= (a^3 + b^3 + c^3) + 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b)$

$+6abc$

Alternatively, this could have been deduced by noting that the terms fall into one of the 3 groups above.

Then there is only 1 way of creating an a^3 term from

$(a+b+c)(a+b+c)(a+b+c)$; namely by choosing the a from each of the 3 brackets.

There are 3 ways of creating an a^2b term: 3[number of ways of choosing the b]$\times$ 1[number of ways of choosing two as from the remaining 2 brackets].

Finally, there are 6 ways of creating an abc term: 3[number of ways of choosing the a]$\times$ 2[number of ways of choosing the b from the remaining 2 brackets]$\times$ 1[number of ways of choosing the c from the remaining bracket].

The final expression then follows by symmetry.

(ii) $(a+b+c)^4 = (a^4+b^4+c^4)$

$+4(a^3b+a^3c+b^3a+b^3c+c^3a+c^3b)$

$+6(a^2b^2+a^2c^2+b^2c^2)+12(a^2bc+b^2ac+c^2ab)$

For the a^2b^2 term etc, there are $\binom{4}{2}=6$ ways of choosing the brackets from $(a+b+c)(a+b+c)(a+b+c)(a+b+c)$

to give a^2, and then just 1 way of obtaining the b^2 term.

For the a^2bc term etc, there are $\binom{4}{2}=6$ ways of choosing the brackets for the a^2 again, multiplied by the 2 ways of choosing brackets for the b and c.

For further investigation: the 'trinomial' expansion of $(a + b + c)^n$ can be shown to be $\sum_{\substack{i,j,k \\ (i+j+k=n)}} \binom{n}{i,j,k} a^i b^j c^k$,

where $\binom{n}{i,j,k} = \frac{n!}{i!j!k!}$

(with a further extension to the 'multinomial' expansion of

$(a_1 + a_2 + \cdots + a_m)^n$)