MAT Problems B - Trigonometry (Sol'ns)

(4 pages; 14/9/17)

(1) Solve $sin\theta = cos4\theta$ for $0 < \theta < \pi$

Solution

$sin\theta = sin(\frac{\pi}{2} - 4\theta)$

Hence $\theta = \frac{\pi}{2} - 4\theta + 2n\pi$ (1) or $\theta = \left(\pi - \left[\frac{\pi}{2} - 4\theta\right]\right) + 2n\pi$ (2)

From (1), $5\theta = \frac{\pi(1+4n)}{2}$, so that $\theta = \frac{\pi(1+4n)}{10}$

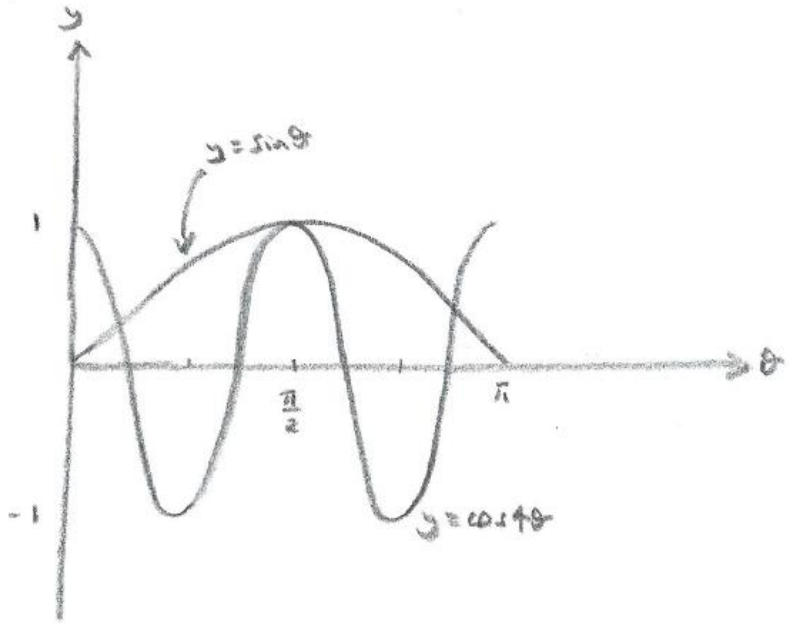
giving $\theta = \frac{\pi}{10}$, $\frac{\pi}{2}$ or $\frac{9\pi}{10}$

From (2), $-3\theta = \frac{\pi(1+4n)}{2}$, so that $\theta = \frac{-\pi(1+4n)}{6}$

giving $\theta = \frac{\pi}{2}$ again

Thus, the solutions are $\theta = \frac{\pi}{10}$, $\frac{\pi}{2}$ or $\frac{9\pi}{10}$

A sketch confirms that these are plausible.



Note: We could attempt $sin\theta = 1 - 2sin^2(2\theta)$

$= 1 - 8sin^2\theta(1 - sin^2\theta)$

giving $8x^4 - 8x^2 - x + 1 = 0$, where $x = sin\theta$

and we can see that $x = 1$ is a root, by the Factor theorem (or by spotting that $\theta = \frac{\pi}{2}$ satisfies the original equation).

However, even if we could solve the remaining cubic, we couldn't guarantee to be able to obtain θ from $x = sin\theta$ without a calculator.

(2) How many solutions does the equation

$sin(2cos(2x) + 2) = 0$ have, for $0 \leq x \leq 2\pi$?

Solution

With $u = 2cos(2x) + 2, 0 \leq x \leq 2\pi \Rightarrow 2(-1) + 2 \leq u \leq 2(1) + 2$

ie $0 \leq u \leq 4$

Then $sinu = 0 \Rightarrow u = 0 \; or \; \pi$

$\Rightarrow \cos(2x) = -1 \; or \; \frac{\pi-2}{2} = \frac{\pi}{2} - 1$

Now making the substitution $w = 2x, 0 \leq w \leq 4\pi$

Referring to the graph of $cosw$,

$cosw = -1$ has 2 solutions (for w), and $cosw = \frac{\pi}{2} - 1$ has 4 solutions; making 6 solutions in total.

As $x = \frac{w}{2}$, there are also 6 solutions for x.

[A variation on the above approach is to say that

$2cos(2x) + 2$ must equal $n\pi$, for suitable integer n

Then, either $n = 0$, with $cos(2x) = -1$,

or $n = 1$, with $cos(2x) = \frac{\pi}{2} - 1$

(no other values of n are consistent with $2cos(2x) + 2$),

as before.]

(3) Assuming that $sin^2\theta + cos^2\theta = 1$, but without using any compound angle results, show that $sin\theta cos\theta \leq \frac{1}{2}$

Solution

$(sin\theta - cos\theta)^2 \geq 0 \Rightarrow sin^2\theta + cos^2\theta - 2sin\theta cos\theta \geq 0$

$\Rightarrow 1 \geq 2sin\theta cos\theta \Rightarrow sin\theta cos\theta \leq \frac{1}{2}$

(4) What is the period of $2\,sin\left(3x+\frac{\pi}{4}\right)+3cos(\frac{2x}{3}-\frac{\pi}{3})$?

Solution

The period T_1 of $2\,sin\left(3x+\frac{\pi}{4}\right)$ satisfies $3T_1 = 2\pi$

[as $2sin\left(3[0]+\frac{\pi}{4}\right) = 2sin\left(2\pi+\frac{\pi}{4}\right)$]; ie $T_1 = \frac{2\pi}{3}$

Similarly for $3cos(\frac{2x}{3}-\frac{\pi}{3}), \frac{2T_2}{3} = 2\pi$, so that $T_2 = 3\pi$

The period of the sum of these functions is the LCM of these two periods; ie 6π.