MAT Problems A - Integers (Sol'ns) (4 pages; 14/9/17)

(1) Show that the product of 4 consecutive positive integers is never a perfect square.

Solution

First of all, $(1)(2)(3)(4) = 24$

$(2)(3)(4)(5) = 120$

$(3)(4)(5)(6) = 360$

$(4)(5)(6)(7) = 840$

and we note that $24 = 5^2 - 1$, $120 = 11^2 - 1, 360 = 19^2 - 1$

& $840 = 29^2 - 1$

So, if we can prove that $n(n+1)(n+2)(n+3) + 1$ is always a perfect square, then we will have the required result.

The sequence $5, 11, 19, 29$ has 1st differences of $6, 8$ & 10, and 2nd differences of 2, and so is quadratic and of the form

$\frac{1}{2}(2)n^2 + an + b$

Setting n equal to 1 & 2 shows that the sequence is $n^2 + 3n + 1$

So, we want to show that

$n(n+1)(n+2)(n+3) + 1 = (n^2 + 3n + 1)^2$

Method 1: Expand both sides

Method 2: Induction

Having shown that the result is true for $n = 1$, we assume that it is true for $n = k$, so that

$k(k+1)(k+2)(k+3) + 1 = (k^2 + 3k + 1)^2$ (A)

We then want to show that, if the result is true for $n = k$, then it will be true for $n = k + 1$;

ie that (B):

$(k+1)(k+2)(k+3)(k+4) + 1 = ([k+1]^2 + 3[k+1] + 1)^2$

By subtracting (A) from (B), this is equivalent to showing (C):

$([k+1]^2 + 3[k+1] + 1)^2 - (k^2 + 3k + 1)^2$

$= (k+1)(k+2)(k+3)(k+4) - k(k+1)(k+2)(k+3)$

LHS of (C) $= \{[k+1]^2 + 3[k+1] + 1 + k^2 + 3k + 1\}$

$\times \{[k+1]^2 + 3[k+1] + 1 - (k^2 + 3k + 1)\}$

$= \{2k^2 + 8k + 6\}\{2k + 4\} = 4(k^2 + 4k + 3)(k + 2)$

$= 4(k+1)(k+3)(k+2)$

whilst RHS of (C) $= (k+1)(k+2)(k+3)\{k + 4 - k\}$,

giving the same expression

Thus the result is true for $n = 1$, and if it is true for $n = k$, then it will be true for $n = k + 1$. Hence it must be true for $n = 2, 3, \ldots$, and therefore all positive integers, by the principle of induction.

(2) Show that numbers of the form $4(n-1)^2 + 2$ can never be one more than a multiple of 3, where n is a positive integer.

Solution

Case 1: $n = 3p$ (where $p \in \mathbb{Z}^+ \; or \; 0$)

$4(n-1)^2 + 2 = 4(3p-1)^2 + 2 \equiv 4(1) + 2 \pmod 3 \equiv 0$

[as $4(3p)^2 + 4(2)(3p)(-1)$ is a multiple of 3]

Case 2: $n = 3p + 1$

$4(n-1)^2 + 2 = 4(3p)^2 + 2 \equiv 2$

Case 3: $n = 3p + 2$

$4(n-1)^2 + 2 = 4(3p+1)^2 + 2 \equiv 4 + 2 \ \equiv 0$

So $4(n-1)^2 + 2$ is always $\equiv 0 \ or \ 2$; ie never one more than a multiple of 3.

(3) Find all positive integer solutions of the equation

$xy - 8x + 6y = 90$

Solution

[Aiming for something of the form $f(x)g(y) = c$, where c is an integer:]

$xy - 8x + 6y = (x+6)(y-8) + 48$,

so that the original equation is equivalent to

$(x+6)(y-8) = 42$

The positive integer solutions are given by:

$x + 6 = 7, y - 8 = 6$

$x + 6 = 14, y - 8 = 3$

$x + 6 = 21 \ y - 8 = 2$

$x + 6 = 42, y - 8 = 1$,

so that the solutions are:

$x = 1, y = 14$

$x = 8, y = 11$

$x = 15, y = 10$

$x = 36, y = 9$

(4) Can n^3 equal $n + 572$ (where n is a positive integer)?

Solution

Rearrange to $n^3 - n = 572$

$n^3 - n = n(n^2 - 1) = n(n - 1)(n + 1)$, and one of these factors must be a multiple of 3; whereas 572 is not a multiple of 3 (since 5+7+2 isn't a multiple of 3); so answer is No.