MAT - Ideas (12 pages; 8/9/18)

(A) Circles

(1) A tangent to a circle is perpendicular to the radius at the point in question.

(2) The perpendicular bisector of a chord passes through the centre.

(3) The angle made by a chord at the centre is twice the angle made at the circumference.

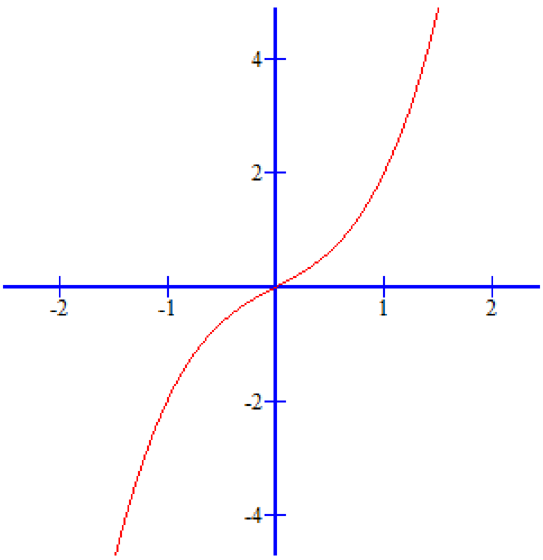
Special case: A triangle with corners on the circumference, and with one side being the diameter, is right-angled (and v.v.)

(B) Cubics

(1) They have one point of inflexion, and have rotational symmetry about this point.

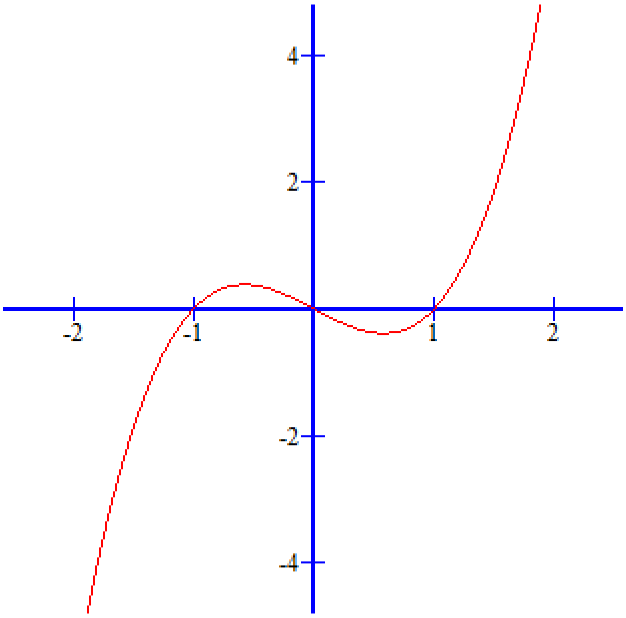
(2) Their shape will be determined by the number of stationary points (0, 1 or 2);

Shape 1: $y = x^3 + x$ (0 stationary points):



Shape 2: $y = x^3$ (1 stationary point)

Shape 3: $y = x^3 - x$ (2 stationary points):



(C) Curve sketching

See also: **Cubics, Transformations, Turning points**

(1) Consider the behaviour of the function as $x \to \infty$.

(2) Is the function increasing or decreasing at a particular point?

(3) Symmetry about the y-axis: $f(-x) = f(x)$.

(4) Vertical asymptotes; eg $y = \frac{x-1}{x+1}$ (asymptote at $x = -1$)

(5) To investigate the intersections of the curves $y = f(x)$ and $y = g(x)$, where $f(x) - g(x) = 0$ is a quadratic in x, consider the discriminant $(b^2 - 4ac)$. This shows whether the curves intersect, touch or don't intersect.

(D) Equations

(1) $f(x) = 0$ may be reformulated as $g(x) = h(x)$, and the points of intersection of $y = g(x)$ & $y = h(x)$ investigated.

(E) Inequalities

(1) Methods for solving eg $\frac{1}{x} < x$

<u>Method 1</u>: Multiply both sides by x^2

<u>Method 2</u>: Treat the cases $x < 0$ and $x > 0$ separately

<u>Method 3</u>: Rearrange as $\frac{1}{x} - x < 0$

<u>Method 4</u>: Sketch $y = \frac{1}{x}$ and $y = x$, and consider points of intersection

(2) Inequalities involving moduli signs

Example A: $|x - 2| > 5$

Method 1

x is more than 5 away from 2, and so has to be either < -3 or > 7

Method 2

$|x - 2| > 5 \Leftrightarrow (x - 2)^2 > 25$ (see (3) Useful devices)

$\Leftrightarrow x^2 - 4x - 21 > 0$

$\Leftrightarrow (x - 7)(x + 3) > 0$

$\Leftrightarrow x < -3$ or $x > 7$

Method 3

[Note: $|a| = a$ when $a \geq 0$ and $-a$ when $a < 0$]

Case 1: $x-2 \geq 0$; ie $x \geq 2$

Then $x-2>5$, so that $x>7$ and $x \geq 2$; ie $x>7$

Case 2: $x-2<0$; ie $x<2$

Then $-(x-2)>5$, so that $-x>3$ and $x<2$; ie $x<-3$

Hence $x<-3$ or $x>7$

Method 4

Draw graphs of $y=|x-2|$ and $y=5$

Example B: $2<|x+3|<7$

Method 1

$\Leftrightarrow$ distance of x from -3 is between 2 and 7

So $-10<x<-5 \; or -1<x<4$

Method 2

$\Leftrightarrow 4<(x+3)^2<49$

$\Leftrightarrow x^2+6x+9-4>0 \; and \; x^2+6x+9-49<0$

$\Leftrightarrow x^2+6x+5>0 \; and \; x^2+6x-40<0$

$\Leftrightarrow (x+5)(x+1)>0 \; and \; (x+10)(x-4)<0$

$\Leftrightarrow x<-5 \; or \; x>-1$ and $-10<x<4$

So $-10<x<-5 \; or -1<x<4$

Method 3

Case 1: $\boldsymbol{x+3\geq 0}$ **(ie** $\boldsymbol{x\geq -3}$)

Then $2<x+3<7$, so that $x>-1\ and\ x<4$

Hence, for case 1, $-1<x<4$

Case 2: $\boldsymbol{x+3<0}$ **(ie** $\boldsymbol{x<-3}$)

Then $2<-(x+3)<7$, so that $-x>5\ and -\ x<10$

So $x<-5\ and\ x>-10$

Hence, for case 2, $-10<x<-5$

Example C: $|x-2|>|x-5|$

Method 1

x has to be further from 2 than from 5

It is equidistant when $x=\frac{7}{2}$, and so has to be $>\frac{7}{2}$

Method 2

$|x-2|>|x-5|\Leftrightarrow(x-2)^2>(x-5)^2$

$\Leftrightarrow -4x+4>-10x+25$

$\Leftrightarrow 6x>21\Leftrightarrow x>\frac{7}{2}$

Method 3

The critical points are $x=2$ and $x=5$

Case 1: $x<2$

$|x-2|>|x-5|\Leftrightarrow -(x-2)>-(x-5)$

$\Leftrightarrow 2>5$; ie no sol'ns

Case 2: $2 \leq x < 5$

$|x-2| > |x-5| \Leftrightarrow x-2 > -(x-5) \Leftrightarrow 2x > 7 \Leftrightarrow x > \frac{7}{2}$

So $2 \leq x < 5$ and $x > \frac{7}{2}$; ie $\frac{7}{2} < x < 5$ is a sol'n.

Case 3: $x \geq 5$

$|x-2| > |x-5| \Leftrightarrow x-2 > x-5 \Leftrightarrow -2 > -5$

So $x \geq 5$ is a sol'n.

So $x > \frac{7}{2}$

Method 4

Draw the graphs of $y = |x-2|$ and $y = |x-5|$

(3) Useful devices

(i) If a & b are ≥ 0, then $a > b \Leftrightarrow a^2 > b^2$ (as $y = x^2$ is an increasing function for $x \geq 0$).

eg $|x-1| > |x+2| \Leftrightarrow (x-1)^2 > (x+2)^2$

(ii) If an expression can be arranged into the form $(a-b)^2$, then this will be non-negative.

(iii) Consider critical values where equality holds.

(F) Integers

(1) Tests for divisibility

(i) sum of digits is multiple of 3 $\Leftrightarrow$ number is multiple of 3 (similarly for 9)

(ii) eg $11 \times 325847 = 3584317$

$3 - 5 + 8 - 4 + 3 - 1 + 7 = 11$, which is a multiple of 11, so 3584317 is a multiple of 11

(2) Consider separately the cases: n even; n odd

(3) Consider factorisations

eg $xy - 8x + 6y = 90$

rearrange to $(x + 6)(y - 8) = 42$

(4) Difference of 2 squares: $m^2 - n^2 = (m + n)(m - n)$

(G) Logarithms

(1) $log_8 2$ is power needed to get from 8 to 2;

in general, $log_a b = c \Leftrightarrow a^c = b$

(2) eg $3 + log_2 25 = 3log_2 2 + 2log_2 5$

(3) $log_a b \; log_b c = log_a c$ or $log_b c = \frac{log_a c}{log_a b}$

Special case: $log_b c = \frac{1}{log_c b}$

(4) As $log_8 8 = 1$ *and* $log_8 64 = 2$, and as $y = log_8 x$ is a concave function ($\frac{dy}{dx}$ is decreasing; ie $\frac{d^2y}{dx^2} < 0$), linear interpolation

$\Rightarrow log_8 \left[\frac{1}{2}(8 + 64)\right] > \frac{1}{2}(1 + 2)$

ie $log_8 36 > \frac{3}{2}$

(5) eg $243 = 3^5 < 2^8 = 256$, so that $5log_2 3 < 8$

and $log_2 3 < \frac{8}{5}$

Also $log_3 4 < \frac{4}{3} \Leftrightarrow 4 < 3^{\frac{4}{3}} \Leftrightarrow 4^3 < 3^4 \Leftrightarrow 64 < 81$

(for a good approximation, the two values (ie 243 & 256 or 64 & 81) should be as close as possible)

(6) eg $log_2 12 = log_2(3 \times 4) = log_2 3 + log_2 4 < \frac{8}{5} + 2 = \frac{18}{5}$,

from (5)

(7) eg $log_{36} 8 = \frac{1}{log_8 36} < \frac{2}{3}$, from (4)

(8) Note that eg $log_a\left(\frac{1}{2}\right) < 0$, so that

$x log_a\left(\frac{1}{2}\right) < 1 \Rightarrow x > \frac{1}{log_a\left(\frac{1}{2}\right)} = -\frac{1}{log_a 2}$

(H) Moduli signs (See also **Inequalities**)

(1) Case by case approach for functions involving moduli signs;

eg for $y = |x - 2| + 1$

Case (i) : $x - 2 \geq 0$

$\Rightarrow y = x - 2 + 1 = x - 1$

Case (ii) : $x - 2 < 0$

$\Rightarrow y = (2 - x) + 1 = 3 - x$

(2) eg for $y = |x - 1| + |x + 2|$,

consider the cases $x \leq -2, -2 < x < 1, x \geq 1$

(3) $y = |x - 2| + 1$ has similarities to y = $(x - 2)^2$ + 1

(4) $\sqrt{x^2} = |x|$ (by convention, $\sqrt{}$ means the positive root - unless zero)

(5) For $y = |f(x)|$, when $f(x) = 0$, there will be a cusp.

Note when sketching the curve that $f'(x_0 + \delta) = -f'(x_0 - \delta)$, but that the shape of the curve may differ significantly away from x_0.

(I) Points of inflexion

[Not specifically in the MAT syllabus, but of relevance to 2nd derivatives.]

(1) A point of inflexion is a turning point of the gradient.

(2) Necessary and sufficient condition is that the 1st non-zero derivative of the function, excluding $\frac{dy}{dx}$, should be of odd order.

(3) $\frac{d^2y}{dx^2} = 0$ is a necessary (but not sufficient) condition for a point of inflexion (e.g. $\frac{d^2y}{dx^2} = 0$ at $x = 0$ for $y = x^4$, but there is no point of inflexion)

(4) $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$ is a sufficient (but not necessary) condition for a point of inflexion (eg $y = x^5$, which has a point of inflexion at at $x = 0$; $\frac{d^3y}{dx^3} = 0$, but $\frac{d^5y}{dx^5} \neq 0$)

(5) At a point of inflexion, the curve changes from being concave ($\frac{d^2y}{dx^2} < 0; eg\ y = lnx$) to convex ($\frac{d^2y}{dx^2} > 0; eg\ y = e^x\ [conve^x]$), or vice-versa.

(6) A point of inflexion need not be a stationary point (ie where $\frac{dy}{dx} = 0$); eg $y = sinx$ at $x = 0$

(J) Polynomials

(1) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

Also, if n is odd:

$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1})$

(2) A polynomial of the form $(x - a)^{2m}(x - b)^{2n+1}$ ... has a turning point at $(a, 0)$ and a point of inflexion at $(b, 0)$, when $m > 0$ & $n > 0$

(K) Quadratics

(1) Use of $b^2 - 4ac$ in situations where there are supposed to be 0, 1 or 2 solutions.

(2) The turning point of a quadratic is midway between its roots.

(L) Sequences & Series

(1) A sequence will be of the form $an^2 + bn + c$, if the 2nd difference is $2a$.

(2) $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$; $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

(M) Transformations

(1) At each stage of a sequence of transformations, only the following substitutions are permissible:

$x \to x + a$; $x \to kx$ (where a & k can be negative)

and similarly for y

(2) Reflection in $x = L$: $y = f(x) \to y = f(2L - x)$

Reflection in $y = M$: $y = f(x) \to 2M - y = f(x)$

(3) The graph of $y = \frac{x-2}{x-1}$ can be obtained by a sequence of transformations.

First of all, $\frac{x-2}{x-1} = 1 - \frac{1}{x-1}$

Starting with $y = \frac{1}{x}$

(i) translation of $\binom{1}{0}$, to give $y = \frac{1}{x-1}$

(ii) reflection in the x axis, to give $y = -\frac{1}{x-1}$

(iii) translation of $\binom{0}{1}$, to give $y = 1 - \frac{1}{x-1}$

(N) Trigonometry

(1) $sin\theta = \cos(90° - \theta)$ (and $cos\theta = \sin(90° - \theta)$)

(2) As $sin\theta = \sin(180° - \theta)$, it may not be possible to determine angles close to 90° using the Sine rule. Either find small angles first, or use Cosine rule.

(3) To solve eg $\sin(2x - 60°) = 0.5$; $0 \leq x \leq 360°$:

Let $u = 2x - 60°$ and note that $-60° \leq u \leq 300°$

(4) $cos^2\theta + sin^2\theta = 1$ (A) & $cos^2\theta - sin^2\theta = cos2\theta$ (B)

can be used to obtain expressions for $cos^2\theta$ and $sin^2\theta$, by adding/subtracting (A) and (B).

(O) Turning Points

(1) A necessary and sufficient condition for a turning point is that the 1st non-zero derivative of the function should be of even order (and ≥ 2).

(2) $\frac{d^2y}{dx^2} \neq 0$ is a sufficient (but not necessary) condition for a turning point (e.g. $\frac{d^2y}{dx^2} = 0$ at $x = 0$ for $y = x^4$)

(3) The turning point of a quadratic is midway between its roots.

(4) To find the turning points of $y = \frac{x^2-2x+2}{x^2-3x-4}$, consider the quadratic $\frac{x^2-2x+2}{x^2-3x-4} = k$, with $b^2 - 4ac = 0$

(to give a quadratic in k).