MAT Paper - Exam Technique and Methods

(15 pages; 23/1/17)

Exam Technique

(1) For the earlier questions in the multiple choice section, some answers are easy to eliminate, and it is not unusual to be able to arrive at the correct answer largely (or entirely) by a process of elimination (though obviously it would be a good check to establish the validity of the correct answer independently).

[See 2015, Q1(A), where the official solutions are based on such a process of elimination (and in fact a 'proper' solution would be too time-consuming in the exam itself).]

For some of the multiple choice questions it may not be necessary to actually establish the answer; often the form of the answer will do.

eg 2009, Q1(C): drawing diagrams can establish that there are 4 real solutions for $-k \leq c \leq k$, for some k

(2) Often inspiration for a particular problem will only come after experimenting; or an important feature of a problem will not become apparent until then.

Examples

(i) Sketching functions

(ii) Listing different cases

(iii) Substituting particular values of variables

(a) MAT 2009, Q1(C), which concerns the number of solutions of $x^4 = (x-c)^2$:

Trying out different values of c shows that the number of solutions changes where the graphs of $y = x^4$ and $y = (x - c)^2$ touch.

(b) MAT 2010, Q1(J), where combinations such as (a) $a > 1 \; or \; b < 1$, (b) $a < 1 \; or \; b < 1$ need to be investigated: choose representative numbers for a & b ; eg (for (a)) $a = 2, b = \frac{1}{2}$

(c) MAT 2009, Q1(D); finding the smallest +ve integer n such that

$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1}n \geq 100$

Experiment with some numbers. It should become clear that n must be odd.

[Alternatively, look for a rearrangement; in this case splitting into

$1 + 3 + 5 + \cdots$ and $-(2 + 4 + 6 + \cdots)$]

(iv) For Q1, the multiple choice answers may give an idea of how to experiment.

(v) Consider extreme cases: See what happens when one of the variables tends to infinity (also part of the standard graph sketching procedure).

(vi) It may be possible to base your answer on something simpler that you have experience of; eg MAT 2008, Q3(ii): In order to sketch $y = 2^{-x^2}$, start with $y = 2^{-x}$

(3) A typical problem (especially for the multiple choice questions) will have a moderately long (and perhaps obvious) approach, but also a quicker (but less obvious) alternative. This alternative approach will often arise from some form of experimentation.

On spotting a possible approach, briefly look for alternatives before embarking on the solution.

(4) The paper doesn't usually involve large amounts of algebra (in contrast to the STEP paper). However, an algebraic slip can be the cause of a question going horribly wrong, so practice at algebraic manipulation may be a good idea.

In order to spot such slips before they do any damage, get into the habit of reading over each line before moving on to the next one.

(5) Just before embarking on a solution, re-read the question. Also re-read it at the end, in case there is an additional task that you have forgotten about.

(6) Look for ways of checking your answer:

(a) substitution of your solution into equations

(b) an alternative approach (which may become apparent at the end, when the problem seems simpler)

(c) reasonableness check; including a possible estimate of a numerical solution

(7) As the aim is obviously to make efficient use of the available time:

(a) Be prepared to leave a question, and come back to it later (especially if the question seems to be ambiguous). Sometimes an idea will come to you whilst working on another question!

(b) Before embarking on a solution, consider how likely it is that it will work, and how much time it will take.

Is there anything that can be done quickly that is likely to throw light on the problem?

eg 2009, Q1(B) ; finding the point on the circle

$x^2 + y^2 + 6x + 8y = 75$ which is closest to the origin:

convert the equation into the form $(x-a)^2 + (y-b)^2 = r^2$

You could deliberately save something for the end of the exam: it is useful to have a relatively straightforward task to complete in the last few minutes, rather than frantically looking through the paper for something to check.

(8) It is easy to overlook the possibility of using a previous part of a question. Any part that is a very simple task can be assumed to be there for the benefit of future parts.

(9) If a topic looks unfamiliar, remember that knowledge outside the syllabus (roughly C1 & C2) is not assumed, so the question should be self-contained and include definitions of new concepts. Usually such questions turn out to be easier than normal, as the candidate is being rewarded for coping with an unfamiliar topic. Typically the first part will turn out to be quite simple.

(10) Clues in the question; eg:

(a) if you are told that $x \neq a$, then the solution may well involve a division by $x - a$

(b) a condition in the form of an inequality may suggest the use of $b^2 - 4ac$ (especially if it involves a squared quantity)

(c) the presence of a $\pm$ sign suggests that a square root may be being taken at some stage

(d) the presence of squares or pairs of brackets suggests the possible use of Difference of Two Squares.

For example, $(k+1)(k+3)$ could be rearranged as

$([k+2]-1)([k+2]+1) = (k+2)^2 - 1$, if necessary

(e) there may be a clue in other parts of the question

Example: MAT 2008, Q3 (iii) involves 2^{2x-x^2} ; the presence of $2^{-(x-c)^2}$ in (iv) suggests that completing the square may help

(11) The last part of a question won't necessarily be any harder than the earlier parts - especially once you have got on the question-setter's wavelength. Also, the last part might simply be the final (easy) stage in establishing an interesting result.

(12) When practising with past papers, carry out a post-mortem on each question that you do - including ones that you get right:

(a) the official solution may reveal an alternative solution that is either better, or could be used in another situation (but consider alternative solutions yourself first)

(b) for Q2 onwards, consider how you could improve the way that you approached the problem, or set out your answer

(13) As will become clear when you do past papers, the multiple choice questions tend to cover the same themes each year (eg finding the expression with the smallest or greatest value).

(14) For Q2 onwards, make sure you give sufficient explanation to make your method clear; eg describing what you are doing.

Also be careful that the logic of your argument is easy to follow.

For example, if your solution contains the statement A (eg "$xy = 0$"), followed by the statement B (eg "$x = 0$"), , there needs to be something to distinguish between the following interpretations:

(i) $A \Rightarrow B$

(ii) B follows from (or is repeated from) an earlier part of the solution, or from information given in the question

(iii) B is always true

(iv) B is a conjecture to be considered

Every line really needs some indication of where it comes from.

Introducing something with the word "consider" is a good way to indicate that you are starting a new line of argument.

(15) For "if and only if" proofs, it may be sufficient (ie acceptable) to indicate that the line of reasoning is reversible (assuming that this is the case); ie by use of the $\Leftrightarrow$ sign at each stage.

(16) For Q2 onwards, look out for possible refinements to the solution; eg:

(a) considering separately any special cases (eg where division by zero would be involved)

(b) the range of values for which a result is valid

Methods

(A) Creating equations

When setting up equations or inequalities:

(i) use k^2 to represent a positive number

(ii) use $2k$ to represent an even number; $2k + 1$ to represent an odd number

Provided the number of equations equals the number of unknowns, a unique solution may be possible. Though note that the question may have been designed so that one variable cancels out, in which case a smaller number of equations may be sufficient. Also, if only a ratio of two variables is required, one less equation is needed. The method of equating coefficients [see below] also provides a way of obtaining more than one piece of information from a single equation.

Ensure that all of the information provided in the question has been used.

Note the important features of any diagram provided, and find equations that incorporate these features.

Example: MAT 2008, Q4: A right-angled triangle has its corners on the circumference of a circle. This means that the hypotenuse of the triangle is a diameter of the circle.

If necessary, create a variable in order to establish an equation; eg representing a length in a diagram. (Sometimes the advantage of creating an equation is that it gives you something to manipulate; ie in order to make further progress.)

(B) Case by case approach

Example 1: Solve $\frac{x^2+1}{x^2-1} < 1$

Case 1: $x^2 - 1 < 0$; Case 2: $x^2 - 1 > 0$

(Once we know whether $x^2 - 1$ is positive or negative, we can multiply both sides of the inequality by it.)

Example 2: MAT 2009, Q1(D); finding the smallest +ve integer n such that

$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1}n \geq 100$:consider n even & n odd

(C) Stationary Points etc

(i) Behaviour of gradient, and stationary points in particular, can help to establish the shape of a curve.

(ii) "greatest/smallest value" etc: consider stationary points

This can also be applied to integer-valued sequences.

[Sometimes a 'completing the square' style argument is possible:

$a + (x - b)^2$ has its smallest value at $x = b$

$a - (x - b)^2$ has its greatest value at $x = b$]

(iii) points of inflexion (turning points of the gradient: sufficient condition is $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$) [not part of the MAT syllabus, but could help with curve sketching]

(D) 0, 1 or 2 solutions

If a unique solution is required, or if there is to be exactly 2 solutions, or no solutions, then this suggests the solving of a quadratic equation and considering the discriminant $b^2 - 4ac$

"4 solutions" could be related to a quadratic in x^2, for example

Example 1

If a straight line is to be a tangent to a circle, solve the simultaneous equations of the line and the circle, to give a quadratic, for which $b^2 - 4ac$ must equal zero.

Example 2

To show that $\frac{g(x)}{f(x)}$ is defined in a certain range, we want to show that $f(x) = 0$ has no solutions.

(E) Listing possibilities

Often it is possible to create a list of possible cases to be considered. It may well be that the list initially seems to be prohibitively long (or even infinite), but a pattern can often emerge - or some simplying feature of the problem may become clear.

Similarly, for irregular series: establish the 1st dozen terms, say; you may not find a definite pattern, but just discover a simplifying feature; eg the terms may all be ±1; or odd & even terms may behave differently.

Example: MAT 2008, Q1(I)

The function S(n) is defined for positive integers n by S(n) = sum of the digits of n. For example, S(723) = 7 + 2 + 3 = 12. Find the sum of S(1)+S(2)+S(3)+...+S(99).

Start by just writing out:

1+2+...+9

+ (1+0)+(1+1)+(1+2)+...+(1+9)

+ (2+0)+(2+1)+(2+2)+...+(2+9)

+ ...

It can be seen that 1+2+...+9 will occur 10 times, and that additionally there are 10 lots of 1, 10 lots of 2 etc.

(F) Reformulating the problem

Examples

(a) To prove that $f(x) = g(x)$, aim to show that $f(x) - g(x) = 0$

(b) Convert an equation into the intersection of 2 curves

Example: MAT 2008, Q1[J], investigating the number of solutions of the equation $(3 + cosx)^2 = 4 - 2sin^8x,\ for\ 0 \leq x < 2\pi$

Consider separately the functions $y = (3 + cosx)^2$ and $y = 4 - 2sin^8x$,

and the maximum/minimum values of these functions; it can be seen that

$(3 + cosx)^2 \geq 4$, whilst $4 - 2sin^8x \leq 4$, which obviously places a limit on the possible solution

(G) Transitional (or 'critical') points

This involves considering the point(s) at which the nature of a problem changes.

Example 1

Solve $\frac{(x-1)(x+2)(x-3)}{(x+1)(x-2)(x+3)} < 0$

Note that the only points at which the sign of the left-hand side can change are at the roots of $(x-1)(x+2)(x-3) = 0$ and at the vertical asymptotes $x = -1, x = 2$ and $x = -3$.

Example 2: MAT 2009, Q1(C), as mentioned under Exam Technique (1) above.

(H) Substitutions

Example 1: For $sin^8x + cos^6x = 1$, let $y = sin^2x$

Example 2 (MAT 2008, Q1[H]) – investigating the circumstances when the equation 9^x - $3^{x+1} = k$ has real solutions.

We have an initial choice between letting $y = 9^x$ and letting $y = 3^x$. As $y = 9^x$ doesn't look very promising when it comes to 3^{x+1}, try $y = 3^x$ first.

A general method is to try to create a quadratic (as here). [Most questions are designed this way, as other types of equations can't generally be solved.]

(I) Equating coefficients

For example, completing the square can be carried out by this method:

If $x^2 + 4x + 5 = (x+a)^2 + b$,

then equating coefficients of $x \Rightarrow 4 = 2a$

and equating coefficients of x^0 (ie the constant term) $\Rightarrow 5 = a^2 + b$

(J) Inequalities

Example: Solve $\frac{x-2}{x-1} < 3$

(i) Multiplying by a square

We would like to just multiply both sides by $x - 1$, but $x - 1$ may or may not be negative (if it is negative we would have to reverse the inequality).

However, $(x-1)^2$ will never be negative.

Thus, provided that $x \neq 1$,

$(x-2)(x-1) < 3(x-1)^2$

However, in some cases, this could lead to a cubic (or higher order) inequality, which might not easily be solved.

(ii) Consider the cases $x - 1 > 0$ & $x - 1 < 0$ separately.

(iii) $\frac{x-2}{x-1} - 3 < 0 \Rightarrow \frac{x-2-3(x-1)}{x-1} < 0 \Rightarrow \frac{1-2x}{x-1} < 0$

Then consider (a) $1 - 2x < 0$ & $x - 1 > 0$, or (b) $1 - 2x > 0$ & $x - 1 < 0$.

(iv) Sketch the graph of $y = \frac{x-2}{x-1}$ and establish the points of intersection with $y = 3$.

(K) Transformations of functions

(i) Translation of $\binom{a}{0} \Rightarrow x$ replaced with $x - a$

(and translation of $\binom{0}{a} \Rightarrow y$ replaced with $y - a$)

(ii) Stretch of factor k in the x-direction (eg if $k = 2$, graph of $y = x^2$ is stretched outwards, so that the x-coordinates are doubled)
$\Rightarrow x$ is replaced with $\frac{x}{k}$

(and stretch of factor k in the y-direction $\Rightarrow y$ is replaced with $\frac{y}{k}$)

(iii) A reflection in the y-axis is the same thing as a stretch of factor -1 in the x-direction.

(iv) Compound transformations: note that, at each stage, only one of the above operations is permissible.

Example: To transform $y = sinx$ to $y = \sin(2x - \frac{\pi}{3})$,

either (a) translate by $\begin{pmatrix} \frac{\pi}{3} \\ 0 \end{pmatrix}$, to give $y = \sin\left(x - \frac{\pi}{3}\right)$, and then stretch by factor $\frac{1}{2}$, to give $y = \sin(2x - \frac{\pi}{3})$,

or (b) stretch by factor $\frac{1}{2}$, to give $y = \sin(2x)$, and then translate by $\begin{pmatrix} \frac{\pi}{6} \\ 0 \end{pmatrix}$, to give $y = \sin\left(2\left[x - \frac{\pi}{6}\right]\right) = \sin(2x - \frac{\pi}{3})$.

(v) Reflection in line $x = L$

Consider a point with x coordinate $L + \delta$. A reflection in the line $x = L$ can be achieved by reflecting in the y-axis, and then translating by $\begin{pmatrix} 2L \\ 0 \end{pmatrix}$. So

$y = f(x)$ becomes $y = f(-x)$, and then $y = f(-[x - 2L]) = f(2L - x)$

Example: A reflection in the line $x = \frac{\pi}{2}$ transforms $y = sinx$ to

$y = \sin(\pi - x)$ (which is the same function, as $y = sinx$ is symmetric about $x = \frac{\pi}{2}$).

Similarly for a reflection in $y = L$.

(L) Logarithms

(i) Change of base

$log_a b . log_b c = log_a c$ (consider eg $a = 2, b = 8, c = 64$)

(ii) Example: Show that $log_5 10 < \frac{3}{2}$

$log_5 10 < \frac{3}{2} \Leftrightarrow 10 < 5^{\left(\frac{3}{2}\right)}$ (as the log function is increasing)

$\Leftrightarrow 10^2 < 5^3 \Leftrightarrow 100 < 125$

(M) Linear interpolation

Example: to find an approximate value for $log_2 3$: $log_2 2 = 1$ & $log_2 4 = 2$; as 3 is halfway between 2 & 4, linear interpolation gives $\frac{1}{2}(1 + 2) = \frac{3}{2}$ as an approximate sol'n (also, because of the shape of $y = 2^x$ (which lies below the straight line joining the points (1,2) & (2,4)), it follows that $log_2 3 > 3/2$).

(N) Use of representative values

Example: MAT 2008, Q1(B)

Which is the smallest of these values?

(a) $log_{10}\pi$ (b) $\sqrt{log_{10}(\pi^2)}$ (c) $(\frac{1}{log_{10}\pi})^3$ (d) $\frac{1}{log_{10}\sqrt{\pi}}$

Pretend that $log_{10}\pi$ is equal to ½ (ie a number between 0 and 1, which is all we can easily say about $log_{10}\pi$). This can provide a provisional idea of the relative magnitudes of the 4 quantities.

See also Exam Technique: (2)(iii)(b).