Notes & Solutions for Q1-5 of the Nov. 2012 MAT Paper

(10 pages; 5/11/16)

(to be read in conjunction with the official solutions)

Q1/A

It is possible to find an expression for $\frac{dy}{dx}$ in terms of x and y, and hence produce the equation of the tangent to a general point on the circle. This can then be compared with (a)-(d), and it transpires that only (b) is consistent with the equation of the tangent.

However, as this is the first question on the paper, a much simpler approach is to be expected; namely just drawing the circle and the lines (or just plugging in numbers into the lines to demonstrate that they include points that are inside the circle).

Q1/B

As a slight variation on the official solution, we can say that

$N = 2^{k+2m+3n} = (2^m)^2(2^n)^2 2^{k+n}$

Then in order to be able to write 2^{k+n} as a square, we need $k + n$ to be even.

Q1/C

Having established that $log_3(9^2) = 2log_3 9 = 2(2) = 4$, comparisons can be made fairly easily with each of the other expressions.

Q1/D

Solution

For this type of question, it is usually possible to eliminate some of the options by making appropriate observations. However, a systematic approach may well be needed in order to obtain a definitive answer. In the process, insight gained into the problem may itself help to eliminate options.

Consider $c = 0$ first (as this is quick to do): As $A(0) = 0.5$, (d) is eliminated.

It's worth briefly looking at this point to see what the differences are between the remaining options. One issue is the gradient at $c = 0$ (whether zero, constant or a maximum). We could also consider the gradient of A(c) at $c = -1$ and $c = 1$. In fact though, without worrying about a formula for A(c), we can deduce enough about the behaviour at $c = 0$:

From the initial diagram, A(c) increases at its greatest rate when $c = 0$, and this agrees with (a) only.

[Alternatively, for $c \leq 0$, $A(c) = \int_{-c}^{1} x + c\, dx = \left[\frac{1}{2}x^2 + cx\right]_{-c}^{1}$

[It isn't essential to do this by integration.]

$= \left(\frac{1}{2} + c\right) - \left(\frac{1}{2}c^2 - c^2\right) = \frac{1}{2}c^2 + c + \frac{1}{2}$

Option (b) is therefore eliminated, as it isn't a quadratic function for

$c \leq 0$; whilst (c) is the wrong-shaped quadratic (being 'n-shaped', rather than 'u-shaped'). Also $A'(c) = c + 1$, which approaches 1 as c approaches 0, and this is inconsistent with (c), which shows a gradient of zero.]

Additional Comments

From the diagram, the rate of change of $A(c)$ is at a maximum at $c = 0$, and this implies a point of inflexion. Since the rate of change of $A(c)$ is not zero at $c = 0$, (c) is ruled out and the answer must be (a).

Q1/G

Solution

[The question here is whether to draw a diagram, or to solve the simultaneous equations algebraically. Some use of a diagram is probably good practice.]

We can observe from $x + y = k$, or $y = k - x$, that only positive values of k will result in this line passing through the 1st quadrant. This eliminates (a) and (d).

From $2x + ky = 4$, or $y = \frac{4}{k} - \frac{2x}{k}$, with $k > 0$, we see that we have two lines with negative gradients crossing the 1st quadrant.

In general, these may or may not intersect in the 1st quadrant, so that it may be necessary to solve the equations algebraically, to see what is going on.

However, we could use the multiple choice options to possibly shed some light on the situation.

Thus, setting $k = 2$, we get the single equation $x + y = 2$, which means that $k = 2$ works, and hence option (b) can be eliminated; leaving just (c).

The alternative algebraic approach is as follows:

$2x + ky = 4,\ x + y = k$

$\Rightarrow 2(k - y) + ky = 4$

$\Rightarrow y(k - 2) = 4 - 2k$

$\Rightarrow y = \frac{4-2k}{k-2} = -2$, provided that $k \neq 2$

But we only want positive solutions.

If $k = 2$, the original equations become $x + y = 2$, which has positive solutions.

Thus positive solutions only exist when $k = 2$

Answer is (c)

Q1/H

First of all, note that we have an equation in x (rather than t), so we are looking for values of x that satisfy the equation.

As $0 < sint < 1 < \pi$ for $0 < t < \pi$, $\sin(sint) > 0$ in that interval. Hence $\int_0^x \sin(sint)\,dt > 0$ for $0 < x < \pi$.

Similarly, $\int_\pi^x \sin(sint)\,dt < 0$ for $\pi < x < 2\pi$.

We just need to confirm that

$\int_\pi^{2\pi} \sin(sint)\,dt = -\int_0^\pi \sin(sint)\,dt$ to be sure that the areas only balance out when $x = 2\pi$ (so that there is only 1 solution).

Let $u = t - \pi$ [so that the limits of the two integrals become the same]

Then $\int_\pi^{2\pi} \sin(sint)\,dt = \int_0^\pi \sin(\sin(u + \pi))\,du$

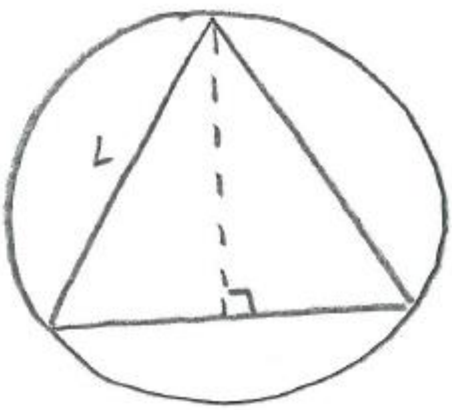
$= \int_0^{\pi} \sin(-sinu)\, du$

[$\sin(u+\pi) = -sinu$, either by the Compound Angle formulae (outside the MAT syllabus), or by translating $sinu$ by $\binom{\pi}{0}$]

$= \int_0^{\pi} -\sin(sinu)\, du$ etc.

Q1/I

From the forms of the multiple choice options, it should be sufficient to obtain an expression for $\frac{A}{P}$.



If we let the side of the equilateral triangle be L, then an alternative method is as follows:

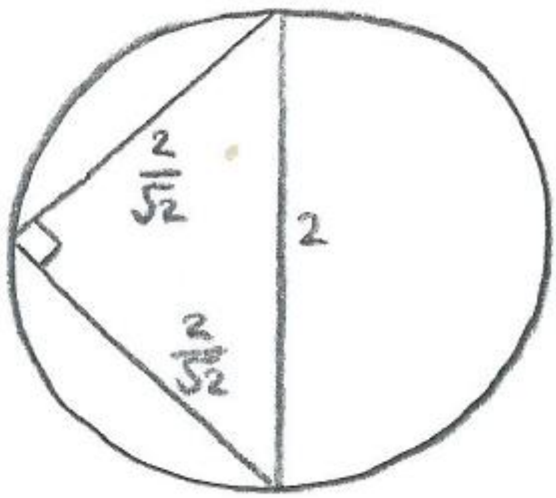
The 'height' of the equilateral triangle is $L\frac{\sqrt{3}}{2}$ (based on the equilateral triangle of side 2 that is used to obtain $sin30°$, $cos30°$ etc), and therefore the area is $\frac{1}{2}L\left(L\frac{\sqrt{3}}{2}\right) = \frac{L^2\sqrt{3}}{4}$.

By a standard result, the centre of mass of the triangle lies $\frac{2}{3}$ of the way along the median from a vertex to the opposite side (a median being the line from a vertex to the midpoint of the opposite side), and by symmetry the centre of mass is the centre of the circle. Thus the radius is $\frac{2}{3}$ of the height of the triangle; ie $\frac{2}{3} \cdot L\frac{\sqrt{3}}{2} = \frac{L}{\sqrt{3}}$

Then, as $2\pi r = 10$, $\frac{A}{P} = \frac{L^2\sqrt{3}}{4} \div 3L = \frac{L}{4\sqrt{3}} = \frac{r}{4} = \frac{5}{4\pi}$

Q1/J

Using the approach in the official solution, it is necessary to spot the required symmetry in order to make any progress (making this quite a difficult question). It is however possible to 'cheat', by observing what happens when $\theta = \frac{\pi}{2}$: the area becomes a semi-circle, together with a right-angled triangle. The area of the latter will be maximised when it is isosceles (by symmetry considerations), giving a total area of $\frac{1}{2}\pi(1)^2 + \frac{1}{2}\left(\frac{2}{\sqrt{2}}\right)\left(\frac{2}{\sqrt{2}}\right) = \frac{\pi}{2} + 1$, which is only consistent with (b). See diagram below.



Note however that this non-rigorous method could be developed to give the symmetry observation needed for the official method, had a written answer been required.

Q2

In (ii), the official sol'ns use the result that $f^2g(x) = gf(x)$ to find other sequences that result in $4x + 4$, but then just state that these are the only such sequences.

In order to arrive at $4x + 4$, there must be exactly two occurrences of g, and a maximum of 4 fs (which occurs for f^4g^2).

So there are ${}^6C_2 = 15$ sequences to consider (the number of ways of positioning the gs).

$ffffgg$ works

$ffgfg$ works

$gffg$ works [$gfffg$ & $gffffg$ can be excluded, as we have already clocked up 4 for the constant term]

ggf works

[$gfgf$ & $gffgf, gfffgf$ don't work]

[$ggff$ & $gfgff, gffgff$ don't work]

[$gfff$ & all further sequences don't work]

Thus the ones that work are $f^4g^2,\ f^2gfg,\ gf^2g,\ g^2f$

As can be seen, by adopting a systematic approach, groups of sequences can be rejected (ie without having to examine each one), so the task isn't as arduous as the figure of 15 combinations might suggest.

For (iv), it is also possible to derive the following pattern of possible combinations of k, i & j:

m, 0, 0

m-1, 0, 2

m-1, 2, 1

m-1, 4, 0

m-2, 0, 4

m-2, 2, 3

m-2, 4, 2

m-2, 6, 1

m-2, 8, 0

...

The numbers of sequences for successive values of k are thus

1, 3, 5, … , with $2r + 1$ sequences for $k = m - r$

The last value for k is $0 = m - m$, so that the total number of sequences is

$1 + 3 + 5 + \cdots + (2m + 1)$

$= \sum_{i=1}^{2m+2} i - 2\sum_{i=1}^{m+1} i = \frac{1}{2}(2m + 2)(2m + 3) - (m + 1)(m + 2)$

$= (m + 1)\{2m + 3 - (m + 2)\} = (m + 1)^2$

Alternative method for the summation:

$1 + (1 + 2 + \cdots + m)$

$+ (2 + 3 + \cdots + (m + 1))$

$= 1 + \frac{1}{2}m(m+1) + \{\frac{1}{2}(m+1)(m+2) - 1\}$

$= \frac{1}{2}(m+1)(m+[m+2]) = (m+1)^2$

[This can be checked for $m = 0$ & 1]

Q3

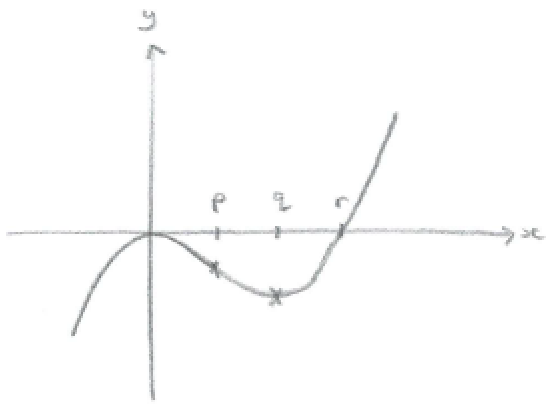
Background information for graphs of cubic functions:

(i) There will always be a point of inflexion, and the curve will have rotational symmetry of order 2 about the PoI.

(ii) Hence, if there are two turning points, the PoI will lie halfway between them.

(iii) Also, if there are 3 real roots (including repeated roots), the x coordinate of the PoI will be the average of the x coordinates of the roots.

So for this question, we can confirm the result in (iii):



$q = 2p$ & $p = \frac{1}{3}(0 + 0 + r) \Rightarrow r = 3p$

We want to show that $r = \frac{3}{2}(x_2 - x_1) = \frac{3q}{2}$, and this follows straightaway from the above results.

[Unfortunately, these results can't be stated without proof: see Pure/Functions/"Point of Inflexion of Cubic".]

Q4

For (iv) & (v), it isn't immediately clear (from the diagram) what situations are possible when the circle touches the vertex of the parabola: (a) is it possible at all? (b) can the parabola touch the circle at points other than the vertex?

There are a couple of points to bear in mind: (i) the rest of the question may shed some light on matters (in this case, (v) makes it clear that the circle can touch the vertex); (ii) it may not be necessary (and there may not be time) to acquire a full understanding of the situation.

Note that, for (iv), any case where the circle touches the vertex can be ruled out as the number of points of contact will then be either 1 or 3.

Given the wording of (v), it doesn't seem to be necessary to establish whether the circle can actually touch the vertex.

Q5

Unusually, parts (ii) & (iii) don't seem to be of any use in tackling parts (v) & (vi). Part (v) can in fact be done more quickly once part (vi) has been answered. Part (vi) can also be answered by going through the P_n until P_8 is reached (based on the relationship between (x_{n+1}, y_{n+1}) and (x_n, y_n) just established).