Notes & Solutions for Q1-5 of the Nov. 2011 MAT Paper

(9 pages; 4/11/16)

(to be read in conjunction with the official solutions)

Q1/A

As all the graphs have a root at $x = -1$, a factorisation can be found by dividing $x^3 - x^2 - x + 1$ by $x + 1$.

In fact (a), (b) and (d) can be eliminated from basic features (large –ve x, y intercept, and $y(1)$, respectively).

Other approaches that could be useful for harder questions involving cubics are:

(i) The Factor Theorem allows us to test factors such as $(x - 1)$.

(ii) $\frac{dy}{dx} = 3x^2 - 2x - 1 = (x - 1)(3x + 1)$ [If (c) is to be true, then $(x - 1)$ must be a factor], so there are stationary points at

$x = 1$ & $-\frac{1}{3}$

Q1/B

Of the 4 possibilities, (c) is the first one that looks feasible.

Then $P^2 - 16A = 4(x^2 + y^2 + 2xy) - 4(4xy) = 4(x - y)^2 \geq 0$

Q1/C

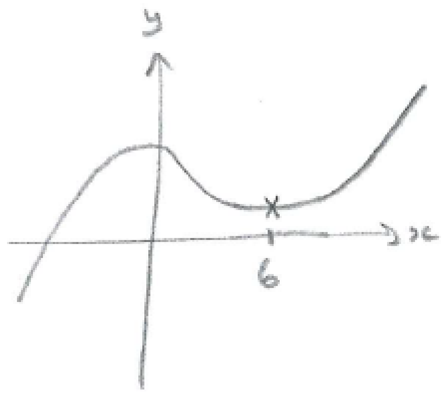
The official sol'ns are missing a " $>$ " sign; ie $-3n^2 + 15n + 8 > 0$

Using the 'official' method, to be absolutely sure that $n = 5$ is the largest value, we can set $\frac{15+\sqrt{X}}{6} = 5$; which gives $X = 225$, so that

$\frac{15+\sqrt{321}}{6} > 5$ (and already shown to be < 5.5)

Alternative method: Let $y = x^3 - 9x^2 + 631$.

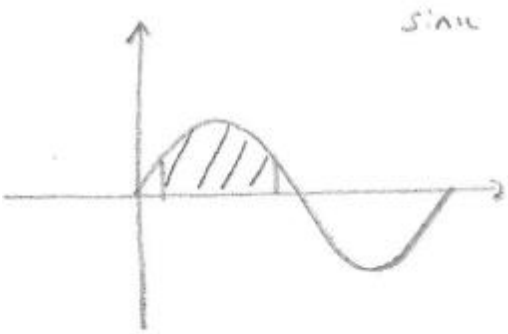
Then $\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 18x = 0 \Rightarrow x = 0\ or\ 3x - 18 = 0; ie\ x = 6$

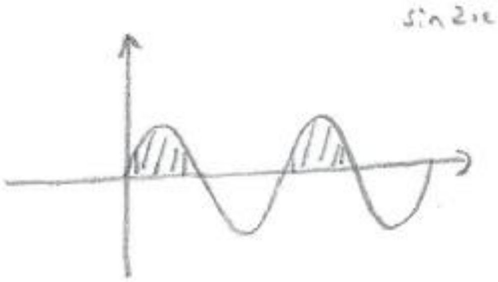


Thus, $x = 5$ is the largest integer, for which $y(x) > y(x+1)$.

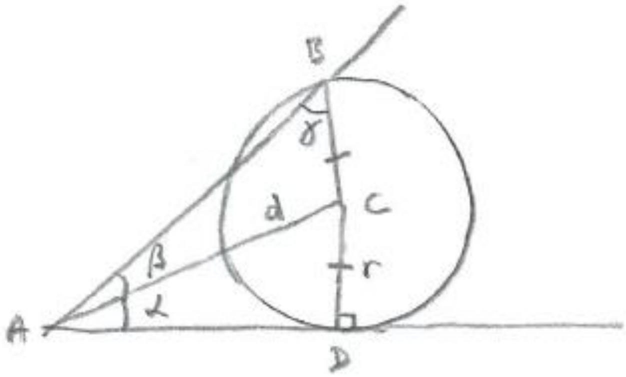
Q1/D

It probably helps to sketch both graphs:





Q1/E



The first step with any question involving circles is to identify any special features. This question involves the centre of the circle, and the natural thing to do is to create any radii that might be useful (since their lengths will be the same). In this case we can draw in CD.

Equations can now be set up, using the Sine rule for ABC and ordinary trigonometry for ACD.

Q1/G

Note that, for $-1 \leq x \leq 1,\ -1 \leq x^2 - 1 \leq 0$

So $f(x^2 - 1) = (x^2 - 1) + 1$ (since the equation of the lefthand sloping part of the graph is $y = x + 1$)

and hence $\int_{-1}^{1} f(x^2 - 1)dx = \int_{-1}^{1} x^2\, dx = \left[\frac{1}{3}x^3\right]_{-1}^{1} = \frac{2}{3}$

Answer: (d)

Q1/H

$x = 8^{log_2 x} - 9^{log_3 x} - 4^{log_2 x} + log_{0.5}0.25$

$\Rightarrow x = 2^{3log_2 x} - 3^{2log_3 x} - 2^{2log_2 x} + 2$

$\Rightarrow x = (2^{log_2 x})^3 - (3^{log_3 x})^2 - (2^{log_2 x})^2 + 2$

$\Rightarrow x = x^3 - x^2 - x^2 + 2$

$\Rightarrow x^3 - 2x^2 - x + 2 = 0$

Unsurprisingly, $x = 1$ is a root,

so $(x - 1)(x^2 + ax - 2) = 0$

and, equating coefficients of x^2, $-2 = a - 1$, so that $a = -1$

(which can be checked by equating coefficients of x)

Then $x^2 - x - 2 = (x + 1)(x - 2)$,

so that there are two positive values of x.

Answer: (c)

Q1/I

Another alternative approach is to let $y = sin^2 x$

Q1/J

The standard approach for this sort of question is to work out the first few values, and see what observations can be made. Note the risk of making a mistake, which is likely to obscure any pattern. A case by case approach can often be adopted, to simplify the problem.

$f(1) = 1$

$f(2) = f(1) = 1$

$f(3) = [f(1)]^2 - 2 = -1$

$f(4) = f(2) = 1$

$f(5) = [f(2)]^2 - 2 = -1$

$f(6) = f(3) = -1$

$f(7) = [f(3)]^2 - 2 = -1$

$f(8) = f(4) = 1$

[Note that working is shown, in order to reveal how each value arises.]

Observations to be made:

(i) all the values seem to be either 1 or -1

(ii) (if they **are** all 1 or -1), then $f(2n+1)$ will always be -1, for $n \geq 1$

(iii) and $f(2n) = f(n)$ won't cause any value to move away from 1 or -1

So, given the starting point $f(1) = 1$, it is clear that all values are indeed either 1 or -1, with odd cases being -1 (apart from

$n = 1$)

Even cases will be -1 if they are of the form $(2^k)(2p+1)$, as division by 2 doesn't change the value, but when an odd number is reached the value becomes -1

Even cases that are powers of 2 will have the value 1, as they all equal $f(1)$

Of the 100 values being added,

$n = 1 \Rightarrow 1$ [1 value]

other odd $\Rightarrow -1$ [49 values]

powers of 2 $\Rightarrow 1$ [6 values]

other even numbers $\Rightarrow -1$ [44 values]

So $f(1) + f(2) + f(3) + \cdots + f(100)$

$= (1 + 6) - (49 + 44) = -86$

Answer: (a)

Q2

(i) $x^3 = 2x + 1 \Rightarrow x^4 = x(2x + 1) = x + 2x^2$

and $x^5 = x(x + 2x^2) = x^2 + 2(2x + 1) = 2 + 4x + x^2$

(ii) $x^{k+1} = x(A_k + B_k x + C_k x^2)$

$= A_k x + B_k x^2 + C_k x^3$

$= A_k x + B_k x^2 + C_k(2x + 1)$

$= C_k + (A_k + 2C_k)x + B_k x^2$

Also $x^{k+1} = A_{k+1} + B_{k+1}x + C_{k+1}x^2$

Equating coefficients:

$A_{k+1} = C_k$; $B_{k+1} = (A_k + 2C_k)$; $C_{k+1} = B_k$

(iii) $D_{k+1} = A_{k+1} + C_{k+1} - B_{k+1}$

$= C_k + B_k - (A_k + 2C_k)$, from (ii)

$= -C_k + B_k - A_k = -D_k$

rtp: $A_k + C_k = B_k + (-1)^k$

ie that $D_k = (-1)^k$

Now, $D_0 = A_0 + C_0 - B_0 = 1 + 0 - 0 = 1$, as $x^0 = 1$

Then $D_{k+1} = -D_k \Rightarrow D_1 = -1;\ D_2 = 1$ ...

and $D_k = (-1)^k$, as required

(iv) $F_k + F_{k+1} = A_{k+1} + C_{k+1} + A_{k+2} + C_{k+2}$ (1)

$F_{k+2} = A_{k+3} + C_{k+3} = C_{k+2} + B_{k+2}$

$= C_{k+2} + (A_{k+1} + 2C_{k+1})$

$= F_k + F_{k+1} - A_{k+2} + C_{k+1}$, from (1)

$= F_k + F_{k+1}$, as required

Q3

(iv) As it isn't immediately obvious which is the best power of x to equate coefficients for, you might consider equating the constant terms instead:

$ma = -b^2c$

Then $c = -\frac{ma}{b^2} = -\frac{(3b^2-1)}{b^2} . \frac{2b^3}{(3b^2-1)} = -2b$

(v) Alternative approach:

$R = \int_b^c m(x-a) - (x^3 - x)dx = \int_b^c -(x-b)^2(x-c)dx$

$= -\left[\frac{1}{3}(x-b)^3(x-c)\right]_b^c + \int_b^c \frac{1}{3}(x-b)^3 dx$ (integrating by Parts)

$= 0 + \left[\frac{1}{12}(x-b)^4\right]_b^c = \frac{(c-b)^4}{12} = \frac{(-3b)^4}{12} = \frac{27b^4}{4}$

The last part of the question then makes it clear that R is to be maximised when $b = -1$ (though this needs to be explained with reference to the diagram) and so $a = \frac{2(-1)^3}{3(-1)^2-1} = -1$ (or directly from the diagram).

Q4

(ii) For the part involving $x^2 + y^2 - 6xy$, just trying critical points such as (0,0) , (1,0) , $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ can help to see what is going on.

(iii) In case it isn't clear, $\frac{1}{\sqrt{5}}(2,1)$ in the official sol'ns is based on the fact that the point (2,1) is a distance $\sqrt{5}$ from the Origin (whilst the edge of Q is a distance 1).

The alternative approach in the official sol'ns can also be set out as follows:

As the radius of the required circle from (2,1) (which touches $x^2 + y^2 = 1$) is $\sqrt{k+5}$, we require $1 + \sqrt{k+5} = \sqrt{5}$ etc

[Note that the point at which the two circles touch has to be on the line from the Origin to (2,1), since the two radii are both perpendicular to the tangent at that point, and therefore are on the same line (and were the tangents to the circles to be different lines, then the circles would intersect, rather than touch - as can be seen by an exaggerated drawing.)]

Q5

(ii) Nothing too profound is expected here. Note that, amongst the possible journeys, there are some that don't reach the goal squares (eg RU); ie they aren't all solutions.

(iii) Alternatively, no. of sol'ns = ${}^{n-1}C_0 + {}^{n-1}C_1 + \cdots + {}^{n-1}C_{n-1}$
(as can be seen from (i))

$= (1+1)^{n-1} = 2^{n-1}$

(iv) This is a rather strange question - making you wonder where you've misinterpreted it - until it becomes clear that even and odd cases behave differently. (It's one of those questions that's worth reading to the end.)

(v) It isn't clear whether it's acceptable to leave the answer as two results, depending on whether n is odd or even. (Given the time constraint, it probably isn't worth coming up with a general formula.)

As an alternative to the $n = 2k \ / \ n = 2k + 1$ approach, you might write $\frac{2}{3}\ [2^{n+\lambda-1} - 1]$, where $\lambda = \frac{1}{2}[(-1)^n + 1]$

as $\lambda = 1$ when n is even, and 0 when n is odd.

However, this isn't a standard approach, so is probably best avoided.