Notes & Sol'ns for Q1-5 of the Nov. 2010 MAT Paper

(10 pages; 28/10/16)

(to be read in conjunction with the official solutions)

Q1/A

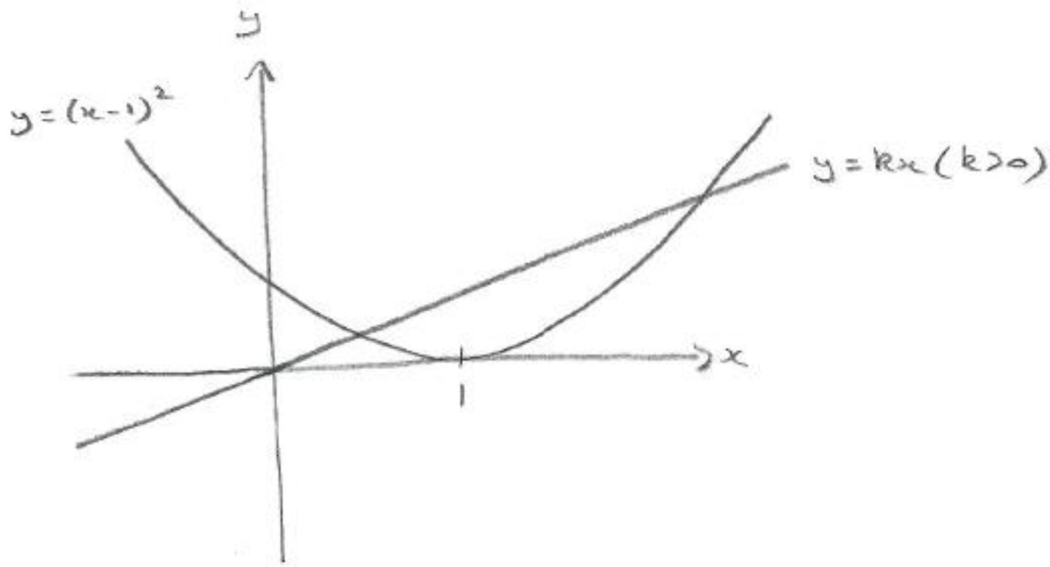
This question demonstrates the classic dilemma with the multiple choice questions: whether to seek to eliminate incorrect answers by means of ad-hoc considerations, or just apply a safe (but possibly longer) method.

Here the discriminant method is the safe one. As an alternative to the working in the official solution, we could say:

$(k+2)^2 - 4 \geq 0 \Rightarrow k^2 + 4k \geq 0 \Rightarrow k(k+4) \geq 0$

and then refer to the graph of $y = k(k+4)$ to establish that $k \geq 0$ or

$k \leq -4$



Alternatively, in order to eliminate (a) and (d), we can simply note that intersections occur for any positive value of k.

For (b), a negative value of k smaller in magnitude than the gradient of $y = (x-1)^2$ at $x = 0$ (namely -2); eg $k = -1$ ensures that the graphs don't meet (since the gradient of $y = (x-1)^2$ increases in magnitude as x becomes more negative, so that the curve $y = (x-1)^2$ moves away from $y = kx$).

As usual though, the main drawbacks of the elimination method are that (a) we don't know in advance which answers we want to be trying to eliminate, and (b) different methods are likely to be needed to eliminate different answers (and we may not succeed in eliminating enough of them).

In case it is needed in a written answer to a long question, the standard symbol for the discriminant is Δ (the same as the symbol for the determinant - which is probably why it is often avoided in A Level textbooks; especially as the two words are often confused anyway).

Q1/D

Maxima occur at $\sqrt{x} = (2k+1)\pi/2$, so that $x = (2k+1)^2\frac{\pi^2}{4}$; ie the gap between x-values increases, so the answer must be (b).

Q1/E

Let $log_2 3 = a$ etc

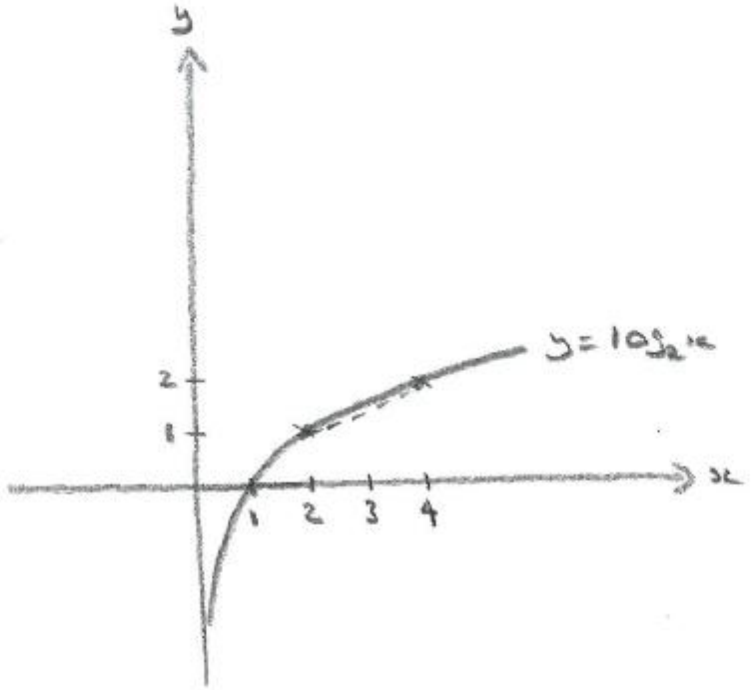
There doesn't seem to be any general procedure for deciding on the relative sizes of $log_p x$ & $log_q y$ (which could be very close). But there are various ad-hoc results that we can use. For example, it is often possible to establish whether $log_p x$ is greater or less than a particular value, as shown below.

In this case, we note first of all that

$1 < a < 2, 1 < b < 2, c < 1$ & $1 < d < 2$, so that (c) can be eliminated.

Also $b = log_4 2^3 = 3log_4 2 = 3\left(\frac{1}{2}\right) = 1.5$

Now, from the diagram below we see that $a = log_2 3 > 1.5$



So we just need to compare $log_2 3$ and $log_5 10$. Having established that $a > 1.5$ and noting that, by linear interpolation,

$d = log_5 10 \approx \frac{(25-10)}{(25-5)} log_5 5 + \frac{(10-5)}{(25-5)} log_5 25 = \frac{15}{20} + \frac{5}{20}(2) = \frac{25}{20} =$ 1.25 ,

it is probably worth comparing $log_5 10$ with 1.5

In fact, we can write $log_5 10 = log_5(5 \times 2) = log_5 5 + log_5 2 =$ $1 + log_5 2$

$< 1 + log_5\sqrt{5} = 1.5$, so that $d < 1.5 < a$ and the answer is therefore (a).

A more general approach though is as follows:

$log_5 10 < \frac{3}{2} \Leftrightarrow 10 < 5^{\frac{3}{2}} \Leftrightarrow 10^2 < 5^3 \Leftrightarrow 100 < 125$

So, for future reference, this gives us a way of comparing $log_p x$ with a fraction, where the numerator and denominator are reasonably small.

Q1/G

It is in fact possible to quickly establish the first 16 values of $f(n)$, and to see that $f(16)$ will be the last occurrence of $f(n) = 16$ (since, for even numbers 2n, $f(2n) = f(n)$, and it can be seen that $f(n) > 8$ beyond $n = 8$; for odd numbers 2n+1, $f(2n + 1) = 4f(n) = 2f(2n)$, so that if $f(2n) \neq 16, f(2n + 1) \neq 16$)

Q1/H

This question can also be approached by initially considering the graph of $y = (x - 1)(x - 2) \dots (x - n)$ and then translating it by $\begin{pmatrix} 0 \\ -k \end{pmatrix}$; so that we are looking for roots of the translated function.

Thus for (c), the function $y = (x - 1)(x - 2) \dots (x - n)$ has roots, and the translated function will also have roots (consider odd and even n separately).

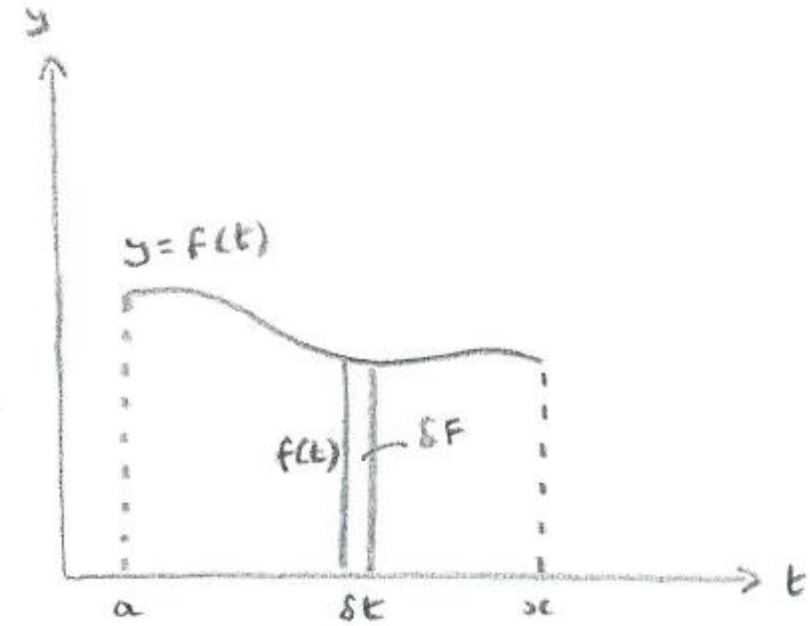
For (d) [to expand on the Official Sol'n], consider a graph with a turning point. A suitable translation (ie value of k) will then bring the turning point onto the x-axis.

Q1/I

This question is based on the 'Fundamental Theorem of Calculus':

If $F(x) = \int_a^x f(t)dt$ [with different notation from the question], then $F'(x) = f(x)$.

It amounts to saying that integration is the opposite of differentiation.



$\delta F \approx f(t)\delta t \Rightarrow \frac{\delta F}{\delta t} \approx f(t)$

$F'(t) = \frac{dF}{dt} = \lim_{\delta t \to 0} \frac{\delta F}{\delta t} = f(t)$

and at $t = x$, $F'(x) = f(x)$

For this question, $\frac{dI}{dA} = 0 \Rightarrow 4 - 2^{a^2} = 0 \Rightarrow a^2 = 2 \Rightarrow a = \sqrt{2}$ (as $a > 0$).

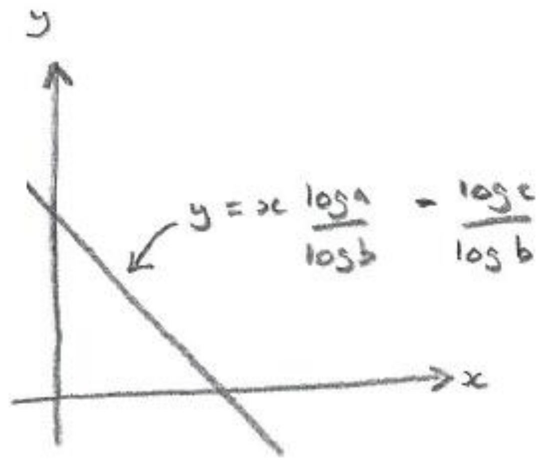
ie Answer is (b)

Q1/J

The approach adopted in the official solution is to explore the implications of firstly $a < 1$ and then $b < 1$, in order to reject 3 of the possible answers, by showing that an infinite number of solutions exists.

The alternative approach mentioned (of taking logs of both sides) is a natural one to apply where indices are involved.

Here we require the gradient of the line $y = x\frac{loga}{logb} - \frac{logc}{logb}$ to be negative (to give a finite number of integer pairs (x, y)), and this is only satisfied by (d). (The log can be to any base of course.)



Q2

For (i), in the official solution, "if $a = 0$ then we have $\sqrt{3} = b/c$ unless

$b = c = 0$" could be expanded as follows:

If $a = 0$, then $b = c\sqrt{3}$

Then either $c = 0$, in which case $b = 0$ also.

Or $c \neq 0$, leading to the contradiction that $\sqrt{3}$ is the rational number b/c.

Hence $a = b = c = 0$.

To expand on the official solution for (iii): If $r < \frac{\sqrt{2}}{2}$, then $N(0.5,0.5,r) = 0$; then, as r is increased, as soon as a point (x, y) is reached, where x and y are integers, exactly 3 other points are reached simultaneously.

The official solution gives the coordinates of these points as $(1 - x, y)$,

$(x, 1 - y)$ and $(1 - x, 1 - y)$.

$(1 - x, y)$, for example, is obtained by reflecting in the line $x = \frac{1}{2}$; in general a reflection of a point in the line $x = a$ is obtained by replacing the x coordinate with $2a - x$ [the result of a reflection in the y-axis, followed by a translation of $\binom{2a}{0}$: $x \rightarrow -x \rightarrow -(x - 2a) = 2a - x$]

Q3

(i) Area OAC < Area of sector OAC

$\Rightarrow \frac{1}{2}(1)^2 sinx < \frac{1}{2}(1)^2 x \Rightarrow sinx < x$

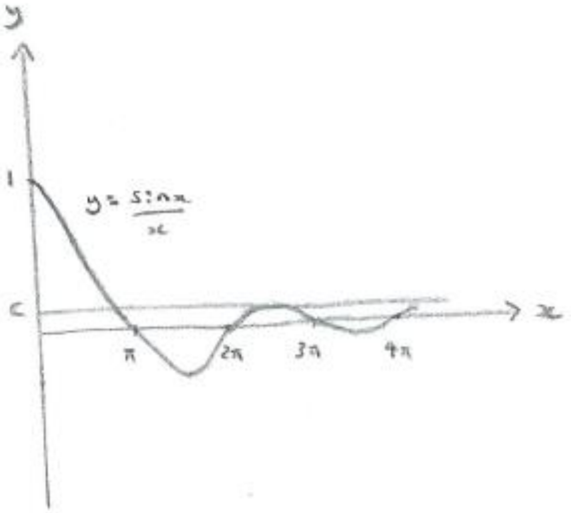
rtp: $xcosx < sinx$ or $x < tanx$ (as $0 < x < \frac{\pi}{2}$ & hence $cosx > 0$)

Area of sector OAC < Area OAB

$\Rightarrow \frac{1}{2}(1)^2 x < \frac{1}{2}(1)tanx$

$\Rightarrow x < tanx$, as required.

(ii) As $x > 0$, $xcosx < sinx < x \Rightarrow cosx < \frac{sinx}{x} < 1$

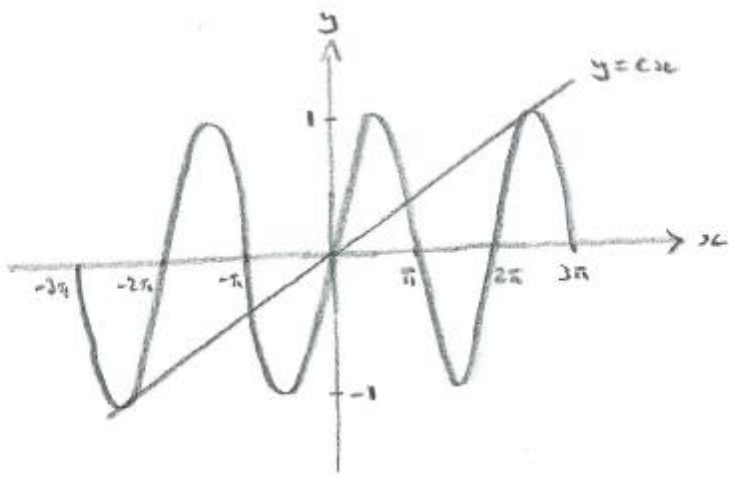


As $cosx \to 1$ as $x \to 0$, $\frac{sinx}{x}$ is trapped between 1 and a number that gets closer to 1, so that $\frac{sinx}{x} \to 1$

[L'H$\hat{o}pital's$ rule: if $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \; or \; \pm\infty$,

then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$]

(iii)



(iv) See (ii). The hump of $y = \frac{sinx}{x}$ in the diagram in (ii) between $x = 2\pi$ & $x = 3\pi$ represents the 1st positive repeated root of $\frac{sinx}{x} = c$, and therefore the 1st positive repeated root of

$sinx = cx$; ie where the graphs of $y = sinx$ & $y = cx$ touch.

(v) X is where $\frac{sinx}{x} = c$; ie the 1st positive maximum of $y = \frac{sinx}{x}$

$\frac{dy}{dx} = 0 \Rightarrow \frac{xcosx - sinx}{x^2} = 0$

$\Rightarrow xcosx - sinx = 0 \Rightarrow x = tanx$; ie $tanX = X$

Q4

(iii) An alternative (though slower) method is to eliminate h from the following 3 eq'ns:

$x^2 + y^2 = 4$

$y = h(3 - x)$ [the eq'n of the tangent]

$y = \frac{x}{h}$ [the eq'n of the radius to the tangent (perpendicular to the tangent), and therefore with gradient $-\frac{1}{(-h)}$]

(v) Consider the form of the answer. This confirms how the required area needs to be split up.

In the official sol'n, "The equation of the line from (i) ..." should presumably read "The equation of the line from (iii) ..."

Here it can be assumed that $\frac{6}{7} < \frac{2}{\sqrt{5}}$, in order for there to be an intersection of the line and circle [we can't have $\frac{6}{7} = \frac{2}{\sqrt{5}}$, obviously]. If we wanted to confirm this, we could compare $\frac{36}{49}$ & $\frac{4}{5}$; eg by establishing that $0.8 \times 49 > 36$

The formula for the area of a triangle with corners $(0,0), (a, b), (c, d)$ can be obtained by considering the matrix transformation $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$: (a, b) is the image of $(1,0)$ and (c, d) is the image of $(0,1)$; the area of the triangle with corners $(0,0), (1,0), (0,1)$ is $1/2$, and the area scale factor is $|ad - bc|$, since $ad - bc$ is the determinant of the matrix (the modulus sign only being needed when the order of the corners becomes reversed in the course of the transformation).

Q5 This question illustrates a couple of standard themes:

(1) The need to try out particular values, in order to discover simplifying features (in this case that large chunks of dates can be ruled out).

(2) Case by case approach; eg "suppose that $d_1 = 3$", which could lead to a convenient contradiction; or else be one branch of the problem to be considered.