Notes & Solutions for Q1-5 of the Nov. 2009 MAT Paper (7 pages; 26/1/17)

(to be read in conjunction with the official solutions)

Q1/A

Tip: look for anything that can be done quickly that is likely to throw some light on the question.

Here, just evaluating the integral is likely to be useful.

Q1/B

Applying the above tip, we can express the equation of the circle in the form $(x-a)^2+(y-b)^2=r^2$.

Tip: Draw a diagram

The required point P will be connected to the origin O by a radius: the line PO must be perpendicular to the tangent at P, otherwise a neighbouring point on the circumference would be closer to O.

Q1/C

This question could also be tackled by drawing graphs of $y=x^4$ and $y=(x-c)^2$, and observing when the number of intersections changes from 4 to some other value.

Alternatively, we could find the values of c for which the curves meet, and where the gradients of $y=x^4$ and $y=(x-c)^2$ are equal (so that the curves are touching).

Q1/D (Ideas & Solution)

Ideas

(i) Consider different cases separately.

(ii) Rearrangement

Solution

In order to simplify the last term on the LHS, we could consider separately the cases of n being even and odd.

With even n, the LHS becomes $1-2+3-4+\cdots-2m$, writing $n=2m$.

By grouping the terms as $(1-2)+(3-4)+\cdots-2m$, we see that this has a negative value.

So n must be odd, and the LHS becomes

$1-2+3-4+\cdots+(2m+1)$, writing $n=2m+1$

And the terms can be grouped to give

$(1-2)+(3-4)+\cdots([2m-1]-2m)+(2m+1)$

$=m(-1)+(2m+1)=m+1$

So we want $m+1\geq 100$, and hence

$n=2m+1\geq 2(99)+1=199$

[A much longer method of summing the LHS - but which is probably worth doing for the practice - is as follows:

$1-2+3-4+\cdots+(2m+1)$

$=\big(1+3+\cdots+(2m+1)\big)-(2+4+\cdots+2m)$

$=\{\sum_{r=1}^{2m+2}r\}-\big(2+4+\cdots+(2m+2)\big)-2\{\sum_{r=1}^{m}r\}$

$=\frac{1}{2}(2m+2)(2m+3)-2\{\sum_{r=1}^{m+1}r\}-m(m+1)$

$= (m+1)(2m+3) - (m+1)(m+2) - m(m+1)$

$= (m+1)(2m+3-m-2-m)$

$= m+1$, as before]

Ans: (c)

Q1/E

Tip: Try out a few values.

Q1/F

Rearranging, we can investigate the intersection of

$y = 3x^4 - 16x^3 + 18x^2$ and $y = -k$

To simplify matters, we can make the substitution $c = -k$

A quick sketch reveals that the answer has to take the form $0 < c < a$, and so $-a < k < 0$; ie the answer is (d), without further calculations (though a could be found by determining stationary points).

Q1/G

$sinx = siny \Rightarrow y = x \pm 2k\pi \; or \; y = \pi - x \pm 2k\pi$

Q1/H

Just involves the sum of a geometric series.

Q1/I

Just involves the Factor theorem.

Q1/J

Rearrange the LHS. The presence of $8y^3$ suggests $(x+2y)^3$.

Having obtained $x + 2y = 2^{10}$, we can simplify matters by writing $u = 2x$ (since x has to be even), to give $u + y = 2^9$.

Then y can take the values $1, 2, \ldots, 2^9 - 1$ (with $x = 2^{10} - 2y$), so that there are $2^9 - 1$ such pairs.

Ans. is (c)

Q2

(i) $x_4 = 2x_3 - x_2 + 1 = 12 - 3 + 1 = 10$

$x_5 = 2x_4 - x_3 + 1 = 20 - 6 + 1 = 15$

(ii) $1 = A + B + C \quad (1)$

$3 = A + 2B + 4C \quad (2)$

$6 = A + 3B + 9C \quad (3)$

Subst. for A from (1) into (2) & (3),

$3 = (1 - B - C) + 2B + 4C \Rightarrow B + 3C = 2 \quad (2a)$

$6 = (1 - B - C) + 3B + 9C \Rightarrow 2B + 8C = 5 \quad (3a)$

Subst. for B from (2a) into (3a),

$2(2-3C)+8C=5 \Rightarrow 2C=1 \Rightarrow C=\frac{1}{2}$

Then (2a) $\Rightarrow B=\frac{1}{2}$ and (1) $\Rightarrow A=0$

(iii) [Assuming that n is supposed to be an integer; $x_{3.5}$, for example, wouldn't be defined]

To find the smallest real number satisfying $\frac{1}{2}x+\frac{1}{2}x^2 \geq 800$:

$x^2+x-1600=0 \Rightarrow x=\frac{-1+\sqrt{1+6400}}{2}$ (as $x>0$)

The smallest integer will then be $\geq \frac{-1+\sqrt{6400}}{2}=\frac{79}{2}$,

and thus the required n is 40

[For a more rigorous proof, we could of course evaluate the quadratic for $n=39$]

(iv) [From the fact that $\frac{x_n}{y_n}$ is supposed to have a limit, we can surmise that a quadratic expression is needed for y_n , given that

x_n has a quadratic form.]

The 1st few terms for y_n are: $1, 5, 11, 19, 29, 41$

The 1st differences are $4, 6, 8, 10, 12,$

and the 2nd differences are all 2.

Therefore, y_n can be represented by a quadratic function of n, where the coefficient of n^2 is $\frac{1}{2}(2)=1$ [this is a standard result, but we are demonstrating that the formula works]

Consider the 1st few terms for y_n-n^2: $0, 1, 2, 3, \ldots$

Thus $y_n-n^2=n-1$,

and $y_n = n^2 + n - 1$

(We know that a quadratic formula exists, and there will only be one such formula that holds for $0, 1, 2$)

[Alternative method: Let $y_n = D + En + Fn^2$, and find D, E & F as in (ii).]

$$\frac{x_n}{y_n} = \frac{\frac{1}{2}n+\frac{1}{2}n^2}{n^2+n-1} = \frac{1}{2}\left(\frac{\frac{1}{n}+1}{1+\frac{1}{n}-\frac{1}{n^2}}\right) \to \frac{1}{2}\left(\frac{1}{1}\right) = \frac{1}{2}$$

[This makes use of a university level theorem that

$lim\frac{f(n)}{g(n)} = \frac{limf(n)}{limg(n)}$, provided that $limf(n)$ & $limg(n)$ are both constants. It seems to be customary to use this theorem without further comment.]

Q3

For (ii), the critical region is $0 < x < 1$. A test value such as $x = \frac{1}{2}$ can be used to examine the behaviour for large n.

For (iv), rearranging the expression found for $\int_0^1 f_n(x)\,dx$ into the form $1 - \cdots$ creates an inequality in n, and comparison of the coefficients of n^2 shows that $A > \frac{3}{4}$ leads to a contradiction.

Once the inequality $B \geq \frac{n}{4(3n-1)}$ is obtained, substitution of $n = 1, 2, \ldots$ reveals the smallest possible value for B.

For a more rigorous proof, the graphs of $y = 12Bn - 4B$ and $y = n$ can be compared. The critical situation occurs when $n = 1$, and we require $12B(1) - 4B \geq 1$

Q4

For (i), the case where $a = 0$ has to be treated separately (as the gradient of the normal becomes infinite).

For (iv), the official solution mentions that there are "many points" which satisfy the condition, but doesn't elaborate, apart from saying that any point on the negative y- axis will do. It would seem that any point not on the y-axis (and not on C) satisfies the required condition. The unique value of a is then determined by finding the point Q for which the normal passes through R.

Q5

For (v), it is a fair bet that the result that $f = n^2$ will be used.