Logic: Implication (6 pages; 8/9/15)

This note covers the following results concerning implication:

(I) $p \Rightarrow q$ is equivalent to $\sim p \vee q$

(II) $p \Rightarrow q$ is equivalent to $\sim q \Rightarrow \sim p$ (contrapositive of $p \Rightarrow q$)

(III) $(p \wedge (p \Rightarrow q)) \Rightarrow q$ ('modus ponens')

(IV) $\sim (p \Rightarrow q)$ is equivalent to $p \wedge \sim q$

Example 1: Let s be the event that it is snowing, and let c be the event that it is cold.

Example 2: Let L be the event that an animal is a lion, and let V be the event that it is a vegetarian.

(1) Consider the proposition $s \Rightarrow c$

(Here we have chosen a proposition that is known to be true, but we could have considered $L \Rightarrow V$, which would be false.)

In the world of logic, only 4 situations are possible (as appear in a truth table):

A: s not true & c not true

B: s not true & c true

C: s true & c not true

D: s true & c true

We are effectively considering an experiment that has already taken place, and one of the above 4 outcomes has occurred (though we may not know which).

By saying $s \Rightarrow c$, we mean that we can be sure that either A, B or D are true (whilst C cannot be, as it clashes with this statement).

Referring to the Venn diagram in Figure 1, the event $A \vee B \vee D$ is the shaded area, which can also be written as $\sim s \vee c$

$s \Rightarrow c$

Figure 1

Thus $s \Rightarrow c \equiv \sim s \vee c$ (result (I)).

(This could also be written $s \Rightarrow c \Leftrightarrow \sim s \vee c$)

So, assuming the truth of $s \Rightarrow c$, if we go outside we will find that one of A, B or D is true (whilst C cannot be true).

$s \Rightarrow c$ can be represented by a truth table:

s	c	$s \Rightarrow c$
0	0	1
0	1	1
1	0	0
1	1	1

The truth table reflects the fact that the proposition $s \Rightarrow c$ might not be true (which would be the case if the combination "snowing and not cold" were able to occur). Of course, the existence of a line in the truth table doesn't mean that the combination in question could arise in practice. For example, in the sahara desert, the 3rd and 4th lines would not occur.

The equivalence of $p \Rightarrow q$ and $\sim p \vee q$ can also be demonstrated using a truth table:

$(p$	$\Rightarrow$	$q)$	$\Leftrightarrow$	$(\sim p$	$\vee$	$q)$
0	1	0	1	1	1	0
0	1	1	1	1	1	1
1	0	0	1	0	0	0
1	1	1	1	0	1	1

If we are given a compound proposition involving $\Rightarrow$ that we wish to simplify, then we may replace any instance of $p \Rightarrow q$ with

$\sim p \vee q$, and then apply the rules of Boolean algebra.

Alternatively, we can use truth tables, as above.

(2) $p \Rightarrow q$ is equivalent to $\sim q \Rightarrow \sim p$ (result (II))

The proposition $\sim q \Rightarrow \sim p$ is termed the '**contrapositive**' of the proposition $p \Rightarrow q$.

Equivalence can easily be demonstrated by employing the equivalence of of $p \Rightarrow q$ and $\sim p \vee q$, as follows:

$p \Rightarrow q \equiv \sim p \vee q$

whilst $\sim q \Rightarrow \sim p \equiv [\sim (\sim q)] \vee (\sim p) \equiv q \vee (\sim p) \equiv \sim p \vee q$

Alternatively, a truth table can be used:

$(p$	$\Rightarrow$	$q)$	$\Leftrightarrow$	$(\sim q$	$\Rightarrow$	$\sim p)$
0	1	0	1	1	1	1
0	1	1	1	0	1	1
1	0	0	1	1	0	0
1	1	1	1	0	1	0

(3) The result (III) $(p \wedge (p \Rightarrow q)) \Rightarrow q$ is sometimes called "modus ponens" (short for "modus ponendo ponens", which is Latin for "the way that affirms by affirming" - for what it's worth).

This result is hopefully self-evident.

Note the distinction between "$\Rightarrow q$ " and "$= q$"

The LHS of (III) $= p \wedge (\sim p \vee q) = (p \wedge \sim p) \vee (p \wedge q)$,

by the Distributive rule

$= p \wedge q$,

of which q is a subset, and so $(p \wedge (p \Rightarrow q)) \Rightarrow q$

(we could also write $(p \wedge (p \Rightarrow q)) \Rightarrow p$, if we wanted to!)

(4) Negation of implication (result (IV))

This is a rather confusing area of logic. We are concerned with the statement $\sim (s \Rightarrow c)$, which can be read as "it is not the case that $s \Rightarrow c$".

From the result that $p \Rightarrow q \equiv \sim p \vee q$,

$\sim (s \Rightarrow c) \equiv \sim (\sim s \vee c) \equiv s \wedge \sim c$, by applying Boolean algebra.

This is telling us that the proposition $\sim (s \Rightarrow c)$ is equivalent to the proposition that it is snowing and not cold. Arguably, however, this doesn't agree with ordinary English usage.

Using a different example, if I always buy one of brands A, B or D (and not C) when I buy some cereals in a supermarket, then the statement that I don't do this would be equivalent, in Logic, to saying that I always buy C, whereas in ordinary English usage, it could arguably mean instead "I don't always buy one of A, B or D; I do something else (unspecified); such as always buying C or D". In other words, in Logic, if $\sim (A \vee B \vee D)$ were true, it wouldn't be possible for me to come home with D, whereas if I now always buy C or D, then it would be!

The Logic interpretation can perhaps be justified by means of the following example:

Suppose that the only numbers in the universe are 1,2,3 & 4. (We could then draw a Venn diagram for the events P(prime) and O(odd), with one number in each of the 4 regions.)

A world where O implies P is equivalent to the world {2,3,4} [1 can't belong to this world because then it wouldn't be the case that O implies P].

The proposition NOT(O implies P) is intended (in Logic) to mean: "we are not in the world where O implies P"; ie not in the world {2,3,4}; in other words, we must be in the world {1}; so NOT(O implies P} is equivalent to O & NOT(P).

The exam question MEI/D2/June 2006, Q1(iii) features this negated implication:

1 (i) Use a truth table to prove $\sim(\sim T \Rightarrow \sim S) \Leftrightarrow (\sim T \wedge S)$. [8]

(ii) Prove that $(A \Rightarrow B) \Leftrightarrow (\sim A \vee B)$ and hence use Boolean algebra to prove that

$$\sim(\sim T \Rightarrow \sim S) \Leftrightarrow (\sim T \wedge S).$$ [5]

(iii) A teacher wrote on a report "It is not the case that if Joanna doesn't try then she won't succeed." He meant to say that if Joanna were to try then she would have a chance of success. By letting T be "Joanna will try" and S be "Joanna will succeed", find the real meaning of what the teacher wrote. [3]

This corresponds to the prime number example, if O becomes NOT(T) and P becomes NOT(S), and to get the 3 marks in (iii) the candidate is expected to say "Joanna will not try and will succeed".

However, this doesn't quite fit in with ordinary English usage. What if the teacher had said "It is not the case that if a number is odd then it is prime"? (which is a true statement) [we're not confined to 1-4 now]. We wouldn't expect the "real meaning" of their statement to be "the number is odd and not prime".

For exam purposes, always use the interpretation according to Logic. (Although the result is often described as 'surprising', there is no disagreement in the literature on the subject about the statement

$\sim (s \Rightarrow c) \equiv s \wedge \sim c$