Logarithms - Exercises (Sol'ns) (5 pages; 9/1/20)

(1*) Show that $\log(4-\sqrt{15}) = -\log(4+\sqrt{15})$

Solution

$\log(4-\sqrt{15}) = -log\left(\frac{1}{4-\sqrt{15}}\right) = -log\left(\frac{4+\sqrt{15}}{16-15}\right) = -\log(4+\sqrt{15})$

[or $\log(4-\sqrt{15}) + \log(4+\sqrt{15}) = \log\{(4-\sqrt{15})(4+\sqrt{15})\}$

$= \log(16-15) = 0$]

(2*) If $k = log_{24}12$, write the following in terms of k:

(a) $log_{24}2$ (b) $log_{24}6$

Solution

(a) $log_{24}2 = log_{24}\left(\frac{24}{12}\right) = log_{24}24 - log_{24}12 = 1-k$

(b) $log_{24}6 = log_{24}\left(\frac{12}{2}\right) = log_{24}12 - log_{24}2 = k-(1-k) =$
$2k-1$

[or $log_{24}6 = log_{24}\left(\frac{24}{4}\right) = log_{24}24 - log_{24}4 = 1 - log_{24}(2^2)$

$= 1 - 2log_{24}2 = 1-2(1-k) = 2k-1$]

(3*) Is $log_2 3 > \frac{3}{2}$?

Solution

$log_2 3 > \frac{3}{2} \Leftrightarrow 3 > 2^{\frac{3}{2}}$ (as $y = 2^x$ is an increasing function)

$\Leftrightarrow 3^2 > 2^3$

So answer is Yes.

(4*) Write $log_2 3$ in terms of logs to the base 10

Solution

Method 1

Standard result: $log_a b\ log_b c = log_a c$

[a is raised to the power of $log_a c$ in order to get to c; alternatively, raise a to the power of $p = log_a b$, to get to b, and then raise b to the power of $q = log_b c$, to get to c; thus $a^p = b\ \ and\ b^q = c$, which gives $(a^p)^q = c$, and hence $a^{pq} = c$, so that $log_a c = pq = log_a b\ log_b c$]

Then $log_b c = \frac{log_{10} c}{log_{10} b}$, so that $log_2 3 = \frac{log_{10} 3}{log_{10} 2}$

Method 2

Set up an equation, as follows:

Let $log_2 3 = x$

[The advantage of creating an equation is that we then have something that can be manipulated.]

$\Rightarrow 3 = 2^x$

$\Rightarrow log_{10} 3 = x log_{10} 2$

$\Rightarrow log_2 3 = x = \frac{log_{10} 3}{log_{10} 2}$

(5*) Simplify $\frac{log_x b}{log_x a}$

Solution

Without loss of generality , we can suppose that $x < a < b$, with $x^p = a$

and $a^q = b$, so that $x^{pq} = b$

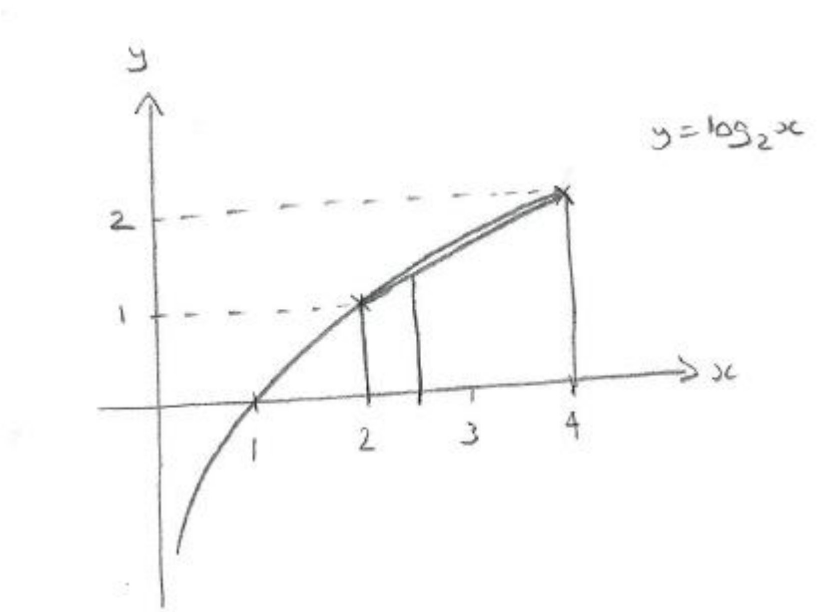
Then $\frac{log_x b}{log_x a} = \frac{pq}{p} = log_a b$

[or $log_x a . log_a b = log_x b$: in terms of powers, p takes you from x to a, and q takes you from a to b; so pq takes you from x to b]

(6*) [Linear interpolation] By approximating the graph of

$y = log_2 x$ by a straight line between $x = 2$ and $x = 4$, find an approximate value for $log_2\left(\frac{5}{2}\right)$

Solution



Approach 1: weighted average

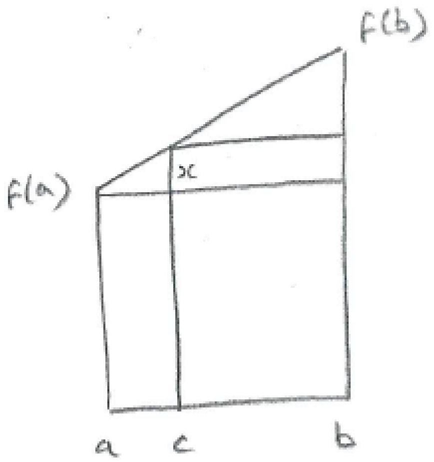
$log_2\left(\frac{5}{2}\right) \approx \left(\frac{4-2.5}{4-2}\right) log_2 2 + \left(\frac{2.5-2}{4-2}\right) log_2 4$

$= (0.75)(1) + (0.25)(2) = 1.25$

Approach 2: similar triangles

Referring to the diagram below (for the general function $f(x)$)

$\frac{x}{c-a} = \frac{f(b)-f(a)}{b-a}$



For our example,

$\frac{x}{2.5-2} = \frac{2-1}{4-2}$,

so that $x = (0.5)(0.5) = 0.25$, and hence $log_2\left(\frac{5}{2}\right) \approx 1 + 0.25 = 1.25$

Approach 3: Equation of line

The gradient of the line is $\left(\frac{f(b)-f(a)}{b-a}\right)$

Then $f(c) \approx f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(c-a)$

In this case, $log_2\left(\frac{5}{2}\right) \approx 1 + \left(\frac{2-1}{4-2}\right)(2.5-2) = 1.25$ again.

Also $f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(c-a)$

$= \left(\frac{1}{b-a}\right)\big((b-a)f(a) + (c-a)f(b) - (c-a)f(a)\big)$

$= \left(\frac{1}{b-a}\right)\big((b-c)f(a) + (c-a)f(b)\big)$,

which is the weighted average approach