# Linear Systems of Differential Equations

(2 pages; 18/11/18)

Example

$\frac{dx}{dt} = 4x - 6y - 9sint$   (1)

$\frac{dy}{dt} = 3x - 5y - 7sint$   (2)

The aim is to find both $x$ and $y$ (the **dependent** variables) as functions of $t$ (the **independent** variable).

To do this we can differentiate (1) wrt $t$, to obtain a 2nd order equation for $x$, and then use (1) and (2) to substitute for $\frac{dy}{dt}$.

Thus, (1)$\Rightarrow \frac{d^2x}{dt^2} = 4\frac{dx}{dt} - 6\frac{dy}{dt} - 9cost$  (3)

Also (2)$\Rightarrow 6\frac{dy}{dt} = 18x - 30y - 42sint$

and then (1) $\Rightarrow 6\frac{dy}{dt} = 18x - 5(4x - 9sint - \frac{dx}{dt}) - 42sint$   (4)

Substituting from (4) into (3) gives:

$\frac{d^2x}{dt^2} = 4\frac{dx}{dt} - (18x - 5(4x - 9sint - \frac{dx}{dt}) - 42sint) - 9cost$  (5)

which gives  $\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = -3sint - 9cost$

This is then solved to give a general solution for $x(t)$, and a general solution for $y(t)$ can be found by substituting for $x(t)$ and $\frac{dx}{dt}$ in (1), having first differentiated $x(t)$ to give $\frac{dx}{dt}$.

In this example, (5) produces the general solution

$x(t) = Ae^{t} + Be^{-2t} + 3cost$  (6)

Then $\frac{dx}{dt} = Ae^t - 2Be^{-2t} - 3sint$

Substituting into (1) then gives

$$Ae^t - 2Be^{-2t} - 3sint = 4(Ae^t + Be^{-2t} + 3cost) - 6y - 9sint$$

which leads to $y = \frac{A}{2}e^t + Be^{-2t} + 2cost - sint$ (7)

Equations (6) & (7) are parametric equations for $x$ and $y$. In simple situations, it may be possible to eliminate $t$, to give a relationship between $x$ and $y$. The resulting graph of $y$ against $x$ is called the **solution curve**.

The system may approach an equilibrium position as $t \to \infty$.