Invariant Points & Lines (6 pages; 25/10/18)

[including Exercises]

(1) An invariant point of a transformation satisfies

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

This may only have the solution $x = y = 0$; ie when the Origin is the only invariant point.

For certain transformations however, there may be a line of invariant points (passing through the Origin).

(2) The line of invariant points for a reflection in the line $y = -x$ is the line itself. This can be verified, as follows:

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow -y = x \;\; \text{and} \;\; -x = y$

These equations are consistent, and give $y = -x$ as the line of invariant points.

(3) An invariant line of a transformation (not to be confused with a line of invariant points) is a line such that any point on the line transforms to a point on the line (not necessarily a different point). A line of invariant points is thus a special case of an invariant line.

We shall see shortly that invariant lines don't necessarily pass through the Origin.

(4) The invariant lines for a reflection in the line $y = -x$ are:

(a) $y = -x$ (the line of invariant points), and

(b) all lines parallel to $y = x$

This can be verified, as follows:

Suppose that an invariant line has the equation $y = mx + c$.

[Strictly speaking, we should also consider lines of the form $x = a$. This possibility is considered for the next example.]

Then the image of a point on this line is determined as follows:

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} -mx - c \\ -x \end{pmatrix}$$

For this image to lie on the line, we require that

$m(-mx - c) + c = -x$

Also, this must hold for all points on the line; ie every x.

Hence, equating coefficients of x: $-m^2 = -1 \Rightarrow m = \pm 1$

and equating the constant terms: $-mc + c = 0$; ie $c(1 - m) = 0$

Then, if $m = 1$, the 2nd condition is satisfied for all values of c.

This gives the lines $y = x + c$ (ie (b) above).

If $m = -1$, the 2nd condition $\Rightarrow c = 0$

This gives the line $y = -x$ ((a) above).

(5) Example

(i) Find the line of invariant points for the shear represented by the matrix $\begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}$

[A necessary and sufficient condition for a shear $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ can be shown to be $ad - bc = 1$ and $a + d = 2$.]

(ii) Find the other invariant lines of the shear, using a matrix method.

Solution

(i) $\begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow 4x - 3y = x$ and $3x - 2y = y$

ie $y = x$

(ii) Suppose that an invariant line has the equation $y = mx + c$

[The possibility of $x = a$ is considered at the end.]

The image of a point on this line is:

$$\begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} 4x - 3mx - 3c \\ 3x - 2mx - 2c \end{pmatrix}$$

For this image to lie on the line, we require that

$m(4x - 3mx - 3c) + c = 3x - 2mx - 2c$

Then, equating coefficients of x: $4m - 3m^2 = 3 - 2m$

$\Rightarrow 3m^2 - 6m + 3 = 0$ or $m^2 - 2m + 1 = 0$,

so that $(m - 1)^2 = 0$ and hence $m = 1$

Equating the constant terms: $-3mc + c = -2c$

$\Rightarrow 3c(1 - m) = 0$, so that either $c = 0$ *or* $m = 1$

Combining the two conditions gives: $m = 1$ and c can take any value; ie the invariant lines are of the form $y = x + c$.

This is to be expected for a shear, where the invariant lines are all the lines parallel to the shear line (which is the line of invariant points).

[Considering lines of the form $x = a$:

$$\begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} a \\ y \end{pmatrix} = \begin{pmatrix} 4a - 3y \\ 3a - 2y \end{pmatrix}$$

For this image to lie on the line, we require $4a - 3y = a$,

and this is only true for $y = a$; ie the single point $\begin{pmatrix} a \\ a \end{pmatrix}$. So there is no invariant **line** of the form $x = a$.]

Exercises

(1) Find the value of k for which the transformation $\begin{pmatrix} 2 & 4 \\ 3 & k \end{pmatrix}$ has a line of invariant points, and find this line.

Solution

Suppose that $\begin{pmatrix} 2 & 4 \\ 3 & k \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

Then $2p + 4q = p$ and $3p + kq = q$

so that $4q = -p$ & $(k - 1)q = -3p$

Hence $\frac{q}{p} = -\frac{1}{4}$ & $\frac{q}{p} = -\frac{3}{k-1}$

so that $-\frac{1}{4} = -\frac{3}{k-1} \Rightarrow k - 1 = 12$ & hence $k = 13$

To find the line:

$$\begin{pmatrix} 2 & 4 \\ 3 & 13 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \Rightarrow 2p + 4q = p \;\&\; 3p + 13q = q$$

so that $4q = -p$ (or $12q = -3p$)

and hence $q = -\frac{p}{4}$

ie the invariant points lie on the line $y = -\frac{x}{4}$

Check: $\begin{pmatrix} 2 & 4 \\ 3 & 13 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

(2) (i) Use a matrix method to find the invariant lines for a reflection in the y-axis.

(ii) Investigate the invariant lines for a reflection in the x-axis.

Solution

(i) Suppose that an invariant line has the equation $y = mx + c$ (noting that lines of the form $x = a$ aren't invariant lines)

The image of a point on this line is:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} -x \\ mx + c \end{pmatrix}$$

For this image to lie on the line, we require that

$m(-x) + c = mx + c$

$\Rightarrow 2mx = 0$

$\Rightarrow m = 0$ (for any value of c), or $x = 0$

ie the invariant lines are $y = c$ and $x = 0$ (the line of invariant points)

(ii) Suppose that an invariant line has the equation $y = mx + c$.

The image of a point on this line is:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} x \\ -mx - c \end{pmatrix}$$

For this image to lie on the line, we require that

$mx + c = -mx - c$

Equating coefficients of x: $m = -m \Rightarrow m = 0$

Equating the constant terms: $c = -c \Rightarrow c = 0$

So we have only found the line $y = 0$ (the line of invariant points).

Now consider lines of the form $x = a$.

This gives $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ y \end{pmatrix} = \begin{pmatrix} a \\ -y \end{pmatrix}$

As this lies on the line $x = a$ for all values of a, the lines $x = a$ are also invariant lines.