

Integration Exercises - Part 3 (Sol'ns) (Hyperbolic Functions)

(12 pages; 6/2/18)

(The constant of integration has been omitted throughout.)

$$(1) \int \operatorname{cosech}^2 x \, dx$$

Solution

Given that $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$, we could investigate $\frac{d}{dx} \coth x =$

$$\frac{d \cosh x}{dx \sinh x} = \frac{\sinh x \cdot \sinh x - \cosh x \cdot \cosh x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\operatorname{cosech}^2 x$$

so that $\int \operatorname{cosech}^2 x \, dx = -\coth x + c$

$$(2) \int \tanh x \, dx \text{ (from 1st principles)}$$

Solution

$$\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} \, dx$$

As $\int \sinh x \, dx = \cosh x$, let $u = \cosh x$

Then $du = \sinh x \, dx$, and $\int \frac{\sinh x}{\cosh x} \, dx = \int \frac{1}{u} \, du = \ln |u| + c$
 $= \ln(\cosh x) + c$

[No moduli signs are needed, as $\cosh x > 0$ for all x]

$$(3) \int \operatorname{sech} x \tanh x \, dx$$

Solution

$$\int \operatorname{sech} x \tanh x \, dx = \int \frac{\sinh x}{\cosh^2 x} \, dx$$

As $\int \sinh x \, dx = \cosh x$, let $u = \cosh x$

Then $du = \sinh x \, dx$, and $\int \frac{\sinh x}{\cosh^2 x} \, dx = \int \frac{1}{u^2} \, du = -\frac{1}{u} + c$

$$= -\operatorname{sech}x + c$$

$$(4) \int \tanh^2 x \, dx$$

Solution

$$\int \tanh^2 x \, dx = \int 1 - \operatorname{sech}^2 x \, dx$$

$$= x - \tanh x + c \quad (\text{from differentiation table})$$

$$(5) \int \operatorname{sech}^2 x \tanh^2 x \, dx$$

Solution

$$\text{As } \int \operatorname{sech}^2 x \, dx = \tanh x, \text{ let } u = \tanh x,$$

$$\text{so that } du = \operatorname{sech}^2 x \, dx,$$

$$\text{and } \int \operatorname{sech}^2 x \tanh^2 x \, dx = \int u^2 \, du = \frac{1}{3}u^3 + c = \frac{1}{3}\tanh^3 x + c$$

$$(6) \int \operatorname{sech}^3 x \tanh x \, dx$$

Solution

$$\text{As we have a power of } \operatorname{sech}x, \text{ consider } \frac{d}{dx} \operatorname{sech}x = -\operatorname{sech}x \tanh x,$$

from (i).

$$\text{Then let } u = \operatorname{sech}x, \text{ so that } du = -\operatorname{sech}x \tanh x \, dx$$

$$\text{and } \int \operatorname{sech}^3 x \tanh x \, dx = -\int u^2 \, du =$$

$$= -\frac{1}{3}u^3 + c = -\frac{1}{3}\operatorname{sech}^3 x + c$$

$$(7) \int \cosh^2 x \, dx$$

Solution

$$\cosh^2 x - \sinh^2 x = 1 \quad \& \quad \cosh^2 x + \sinh^2 x = \cosh 2x,$$

$$\text{so that } \cosh^2 x = \frac{1}{2}(1 + \cosh 2x),$$

$$\text{and } \int \cosh^2 x \, dx = \frac{1}{2} \int 1 + \cosh 2x \, dx = \frac{x}{2} + \frac{1}{4} \sinh 2x + c$$

$$(8) \int \cosh^3 x \, dx$$

Solution

$$\int \cosh^3 x \, dx = \int \cosh x (1 - \sinh^2 x) \, dx$$

$$\text{Let } u = \sinh x, \text{ so that } du = \cosh x \, dx,$$

$$\text{and } \int \cosh x (1 - \sinh^2 x) \, dx = \int 1 - u^2 \, du$$

$$= u - \frac{1}{3} u^3 + c = \sinh x - \frac{1}{3} \sinh^3 x + c$$

$$(9) \int \operatorname{sech} x \, dx$$

Solution

Method 1

$$\int \operatorname{sech} x \, dx = \int \frac{\cosh x}{\cosh^2 x} \, dx = \int \frac{\cosh x}{1 + \sinh^2 x} \, dx$$

$$\text{Let } u = \sinh x, \text{ so that } du = \cosh x \, dx$$

$$\text{Then } \int \frac{\cosh x}{1 + \sinh^2 x} \, dx = \int \frac{1}{1 + u^2} \, du$$

$$= \arctan(\sinh x) + c$$

Method 2

$$\int \operatorname{sech} x \, dx = \int \frac{2}{(e^x + e^{-x})} \, dx = 2 \int \frac{e^x}{e^{2x} + 1} \, dx$$

$$\text{Let } u = e^x, \text{ so that } du = e^x \, dx$$

$$\begin{aligned} \text{and } 2 \int \frac{e^x}{e^{2x}+1} dx &= 2 \int \frac{1}{u^2+1} du \\ &= 2 \arctan(e^x) + c' \end{aligned}$$

To show that $\arctan(\sinh x) + c = 2 \arctan(e^x) + c'$:

Let $\theta = \arctan(\sinh x)$ and $\phi = 2\arctan(e^x)$

We need to show that θ & ϕ differ by a constant.

$$\tan \theta = \sinh x \quad \& \quad \tan\left(\frac{\phi}{2}\right) = e^x$$

$$\text{Let } t = \tan\left(\frac{\phi}{2}\right)$$

$$\text{Then } \sinh x = \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}\left(t - \frac{1}{t}\right) = \frac{t^2-1}{2t}$$

[We can now use the standard right-angled triangle with sides

$1 - t^2, 2t$ & $1 + t^2$: $\tan \phi = \frac{2t}{1-t^2}$ & the hypotenuse follows from Pythagoras.]

$$\text{So } \sinh x = -\tan\left(\frac{\pi}{2} - \frac{\phi}{2}\right) = \tan\left(\frac{\phi}{2} - \frac{\pi}{2}\right)$$

Thus $\tan\left(\frac{\phi}{2} - \frac{\pi}{2}\right) = \tan \theta$, and so $\frac{\phi}{2} - \frac{\pi}{2}$ & θ differ by a multiple of π ; ie θ & ϕ differ by a constant, as required.

$$(10) \int \frac{1}{\sqrt{4x^2-9}} dx$$

Solution

$$\int \frac{1}{\sqrt{4x^2-9}} dx = \frac{1}{2} \int \frac{1}{\sqrt{x^2-\left(\frac{3}{2}\right)^2}} dx$$

$$= \frac{1}{2} \operatorname{arcosh}\left(\frac{x}{\left(\frac{3}{2}\right)}\right) + C = \frac{1}{2} \operatorname{arcosh}\left(\frac{2x}{3}\right) + C$$

$$(11) \text{ (a) } \int \cosh^{-1}x \, dx \quad \text{(b) } \int \sinh^{-1}x \, dx \quad \text{(c) } \int \tanh^{-1}x \, dx$$

Solution

$$\begin{aligned} \text{(a) By Parts, } \int \cosh^{-1}x \, dx &= x \cosh^{-1}x - \int x \frac{1}{\sqrt{x^2-1}} \, dx \\ &= x \cosh^{-1}x - \frac{1}{2} \int \frac{2x}{\sqrt{x^2-1}} \, dx \\ &= x \cosh^{-1}x - \left(\frac{1}{2}\right) \frac{\sqrt{x^2-1}}{\left(\frac{1}{2}\right)} \\ &= x \cosh^{-1}x - \sqrt{x^2-1} \end{aligned}$$

$$\begin{aligned} \text{(b) By Parts, } \int \sinh^{-1}x \, dx &= x \sinh^{-1}x - \int x \frac{1}{\sqrt{x^2+1}} \, dx \\ &= x \sinh^{-1}x - \frac{1}{2} \int \frac{2x}{\sqrt{x^2+1}} \, dx \\ &= x \sinh^{-1}x - \left(\frac{1}{2}\right) \frac{\sqrt{x^2+1}}{\left(\frac{1}{2}\right)} \\ &= x \sinh^{-1}x - \sqrt{x^2+1} \end{aligned}$$

$$\begin{aligned} \text{(c) By Parts, } \int \tanh^{-1}x \, dx &= x \tanh^{-1}x - \int x \frac{1}{1-x^2} \, dx \\ &= x \tanh^{-1}x + \frac{1}{2} \int \frac{-2x}{1-x^2} \, dx \\ &= x \tanh^{-1}x + \frac{1}{2} \ln(1-x^2) \end{aligned}$$

$$(12) \int \frac{1}{(x^2-9)^{\frac{3}{2}}} \, dx$$

Solution

Let $x = 3\cosh u$, so that $dx = 3\sinh u \, du$

$$\text{Then } I = \int \frac{3\sinh u}{27\sinh^3 u} \, du = \frac{1}{9} \int \operatorname{cosech}^2 u \, du$$

As $\frac{d}{du} \tanh u = \operatorname{sech}^2 u$, we can investigate $\frac{d}{du} \coth u$:

$$\frac{d}{du} \frac{\cosh u}{\sinh u} = \frac{\sinh u \sinh u - \cosh u \cosh u}{\sinh^2 u} = -\frac{1}{\sinh^2 u} = -\operatorname{cosech}^2 u$$

$$\text{Thus } I = -\frac{1}{9} \coth u = -\frac{1}{9} \frac{\cosh u}{\sqrt{\cosh^2 u - 1}} = -\frac{1}{9} \frac{\frac{x}{3}}{\sqrt{\frac{x^2}{9} - 1}} = -\frac{1}{9} \frac{x}{\sqrt{x^2 - 9}}$$

Alternative method

Let $x = 3 \sec u$, so that $dx = 3 \sec u \tan u \, du$

$$\begin{aligned} \text{Then } I &= \int \frac{3 \sec u \tan u}{27 \tan^3 u} \, du = \frac{1}{9} \int \frac{1/\cos u}{\left(\frac{\sin u}{\cos u}\right)^2} \, du = \frac{1}{9} \int \frac{\cos u}{\sin^2 u} \, du \\ &= \frac{1}{9} \frac{1}{\sin u} (-1) = -\frac{1}{9} \frac{1}{\sqrt{1 - \cos^2 u}} = -\frac{1}{9} \frac{1}{\sqrt{1 - (3/x)^2}} = -\frac{1}{9} \frac{x}{\sqrt{x^2 - 9}} \end{aligned}$$

$$(13) \int \sqrt{4 + x^2} \, dx$$

Solution

Let $x = 2 \sinh u$, so that $dx = 2 \cosh u \, du$

$$\text{and } I = 2 \int \sqrt{1 + \sinh^2 u} (2 \cosh u) \, du = 4 \int \cosh^2 u \, du$$

$$= 2 \int 1 + \cosh(2u) \, du = 2 \left(u + \frac{1}{2} \sinh(2u) \right)$$

$$= 2 \sinh^{-1} \left(\frac{x}{2} \right) + 2 \sinh u \cosh u$$

$$= 2 \sinh^{-1} \left(\frac{x}{2} \right) + x \sqrt{\left(\frac{x}{2} \right)^2 + 1}$$

$$= 2 \sinh^{-1} \left(\frac{x}{2} \right) + x \sqrt{\left(\frac{x}{2} \right)^2 + 1}$$

$$= 2 \sinh^{-1} \left(\frac{x}{2} \right) + \frac{1}{2} x \sqrt{x^2 + 4}$$

[Note: The alternative substitution $x = 2\tan u$, leads to a much more complicated expression.]

$$(14) \int \sqrt{\frac{x}{1+x}} dx$$

Solution

Let $x = \sinh^2 u$, so that $dx = 2\sinh u \cosh u du$

$$\text{and } I = \int \frac{\sinh u}{\cosh u} (2\sinh u \cosh u) du$$

$$= 2 \int \sinh^2 u du = \int (\cosh(2u) - 1) du$$

$$= \frac{1}{2} \sinh(2u) - u$$

$$= \sinh u \cosh u - u$$

$$= \sqrt{x(1+x)} - \sinh^{-1}(\sqrt{x})$$

$$(15) \int \sqrt{4x^2 - 1} dx$$

Solution

Let $2x = \cosh u$, so that $2 dx = \sinh u du$

$$\text{and } I = \frac{1}{2} \int \sinh u \cdot \sinh u du = \frac{1}{4} \int \cosh(2u) - 1 du$$

$$= \frac{1}{8} \sinh(2u) - \frac{1}{4} u = \frac{1}{4} \sinh u \cosh u - \frac{1}{4} \cosh^{-1}(2x)$$

$$= \frac{1}{2} x \sqrt{4x^2 - 1} - \frac{1}{4} \cosh^{-1}(2x)$$

$$(16) \int x^2 \sqrt{1+x^2} dx$$

Solution

Let $x = \sinh u$, so that $dx = \cosh u du$

$$\text{and } I = \int \sinh^2 u \cosh u \cosh u du$$

$$= \int \sinh^2 u (\sinh^2 u + 1) du$$

[A reduction formula could be found for $\int \sinh^4 u \, du$ at this point, as an alternative to the following.]

$$= \int \frac{1}{2} [\cosh(2u) - 1] \left\{ \frac{1}{2} [\cosh(2u) - 1] + 1 \right\} du$$

$$= \frac{1}{4} \int [\cosh(2u) - 1][\cosh(2u) + 1] du$$

$$= \frac{1}{4} \int \cosh^2(2u) - 1 \, du$$

$$= \frac{1}{4} \int \frac{1}{2} [(\cosh(4u) + 1) - 1] \, du$$

$$= \frac{1}{32} \sinh(4u) - \frac{u}{8}$$

$$= \frac{1}{16} \sinh(2u) \cosh(2u) - \frac{u}{8}$$

$$= \frac{1}{8} \sinh u \cosh u (\cosh^2 u + \sinh^2 u) - \frac{u}{8}$$

$$= \frac{1}{8} x \sqrt{x^2 + 1} (1 + 2x^2) - \frac{1}{8} \sinh^{-1} x$$

$$(17) \int \frac{1}{\sqrt{x(x+1)}} dx$$

Solution

$$x(x+1) = x^2 + x = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}$$

$$\text{So } I = \int \frac{1}{\sqrt{\left(x + \frac{1}{2}\right)^2 - \frac{1}{4}}} dx$$

$$= \cosh^{-1} \left(\frac{x + \frac{1}{2}}{\frac{1}{2}} \right) = \cosh^{-1}(2x + 1)$$

$$(18) \int \frac{x^2}{\sqrt{x^6-1}} dx$$

Solution

Noting that $\int x^2 dx = \frac{1}{3}x^3$, and that $x^6 = (x^3)^2$, let $u = x^3$, so that $du = 3x^2 dx$

$$\text{and } I = \frac{1}{3} \int \frac{1}{\sqrt{u^2-1}} du = \frac{1}{3} \cosh^{-1}(x^3)$$

$$(19) \int_0^4 \frac{4x+1}{\sqrt{x^2+9}} dx$$

Solution

$$I = 2 \int_0^4 \frac{2x}{\sqrt{x^2+9}} dx + \left[\operatorname{arsinh} \left(\frac{x}{3} \right) \right]_0^4$$

$$= 2 \left[\frac{\sqrt{x^2+9}}{\left(\frac{1}{2}\right)} \right]_0^4 + \operatorname{arsinh} \left(\frac{4}{3} \right)$$

$$= 4(5 - 3) + \ln \left(\frac{4}{3} + \sqrt{\left(\frac{4}{3}\right)^2 + 1} \right)$$

$$= 8 + \ln \left(\frac{1}{3} (4 + 5) \right)$$

$$= 8 + \ln 3$$

$$(20) \int_0^1 \sqrt{16x^2 + 9} dx$$

Solution

Let $4x = 3\sinh u$, so that $4dx = 3\cosh u du$

$$I = 3 \int_0^1 \cosh u \cdot \frac{3}{4} \cosh u du$$

$$= \frac{9}{4} \int_0^1 \frac{1}{2} (1 + \cosh 2u) du$$

[since $\cosh^2 u - \sinh^2 u = 1$ & $\cosh^2 u + \sinh^2 u = \cosh 2u$]

$$\begin{aligned}
&= \frac{9}{8} \left[u + \frac{1}{2} \sinh 2u \right]_0^1 \\
&= \frac{9}{8} \left(1 + \frac{1}{4} (e^2 - e^{-2}) \right) = \frac{9}{8} + \frac{9}{32} (e^2 - e^{-2})
\end{aligned}$$

$$(21) \int \sqrt{\frac{x+1}{x}} dx$$

Solution

Let $x = \sinh^2 y$ [so that $x + 1 = \cosh^2 y$]

Then $dx = 2 \sinh y \cosh y dy$

and $I = \int \frac{\cosh y}{\sinh y} \cdot 2 \sinh y \cosh y dy$

$$= 2 \int \cosh^2 y dy$$

$$= \int 1 + \cosh 2y dy$$

$$= y + \frac{1}{2} \sinh 2y$$

$$= \operatorname{arsinh}(\sqrt{x}) + \sinh y \cosh y$$

$$= \operatorname{arsinh}(\sqrt{x}) + \sqrt{x(x+1)}$$

$$(22) \int \frac{1}{(2x^2+3)^{\frac{3}{2}}} dx$$

Solution

Method 1

Let $2x^2 = 3 \sinh^2 y$, so that $4x dx = 6 \sinh y \cosh y dy$

$$\text{Then } I = \int \frac{1}{3^{\frac{3}{2}} (\sinh^2 y + 1)^{\frac{3}{2}}} \frac{6 \sinh y \cosh y dy}{4x}$$

$$= \frac{3}{2} \int \frac{\sinh y \cosh y}{3^{\frac{3}{2}} \cosh^3 y \sinh y \sqrt{\frac{3}{2}}} dy$$

$$= \sqrt{\frac{3}{2}} \int \frac{\operatorname{sech}^2 y}{3^{\frac{3}{2}}} dy$$

$$= 3^{-1} 2^{-\frac{1}{2}} \operatorname{tanh} y$$

$$= \frac{1}{3\sqrt{2}} \operatorname{tanh} y$$

As $2x^2 = 3\sinh^2 y$, $\cosh^2 y = \frac{2}{3}x^2 + 1$,

so that $\operatorname{sech}^2 y = \frac{1}{\frac{2}{3}x^2 + 1} = \frac{3}{2x^2 + 3}$

and $-\operatorname{tanh}^2 y = \operatorname{sech}^2 y - 1$ [applying Osborn's rule]

so that $\frac{1}{3\sqrt{2}} \operatorname{tanh} y = \frac{1}{3\sqrt{2}} \sqrt{1 - \frac{3}{2x^2 + 3}}$

$$= \frac{1}{3\sqrt{2}} \sqrt{\frac{2x^2}{2x^2 + 3}} = \frac{x}{3\sqrt{2x^2 + 3}}$$

Method 2

Let $2x^2 = 3\tan^2 \theta$, so that $4x dx = 6\tan \theta \sec^2 \theta d\theta$

Then $I = \int \frac{1}{3^{\frac{3}{2}}(\tan^2 \theta + 1)^{\frac{3}{2}}} \frac{6\tan \theta \sec^2 \theta}{4x} d\theta$

$$= \frac{3}{2} \int \frac{1}{3^{\frac{3}{2}} \sec^3 \theta} \frac{\tan \theta \sec^2 \theta}{\tan \theta \sqrt{\frac{3}{2}}} d\theta$$

$$= 3^{-1} 2^{-\frac{1}{2}} \int \cos \theta d\theta$$

$$= \frac{1}{3\sqrt{2}} \sin \theta$$

As $2x^2 = 3\tan^2 \theta$, $\sec^2 \theta = \frac{2}{3}x^2 + 1$,

$$\text{so that } \cos^2 \theta = \frac{1}{\frac{2}{3}x^2+1} = \frac{3}{2x^2+3}$$

$$\begin{aligned} \text{and } \frac{1}{3\sqrt{2}} \sin \theta &= \frac{1}{3\sqrt{2}} \sqrt{1 - \frac{3}{2x^2+3}} \\ &= \frac{1}{3\sqrt{2}} \sqrt{\frac{2x^2}{2x^2+3}} = \frac{x}{3\sqrt{2x^2+3}} \end{aligned}$$

$$(23) \int \operatorname{cosech}(3x) \operatorname{coth}(3x) dx$$

Solution

$$\text{Note that } \frac{d}{dx} (\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

So suppose that the solution is of the form $A \operatorname{cosech}(3x)$.

$$\text{Then } \frac{d}{dx} (A \operatorname{cosech}(3x)) = A \frac{d}{dx} ((\sinh(3x))^{-1})$$

$$= A(-1)(\sinh(3x))^{-2} (3 \cosh(3x))$$

$$= -3A \operatorname{coth}(3x) \operatorname{cosech}(3x)$$

$$\text{So we require } A = -\frac{1}{3}$$

$$\text{and } I = -\frac{1}{3} \operatorname{cosech}(3x)$$