Integration Exercises - Part 1 (Sol'ns) (8 pages; 7/11/17)

(Note: The constant of integration has been omitted throughout.)

(1) $I = \int sinx\ cosx\ dx$

Solution

Method 1 (rearrangement)

$\mathrm{I} = \frac{1}{2}\int sin2x\ dx = \frac{1}{2}.\frac{1}{2}(-cos2x) = -\frac{1}{4}cos2x$

Method 2 (substitution)

$Let\ u = sinx,\ \ so\ that\ du = cosx\ dx$

$I = \int u\ du = \frac{1}{2}u^2 = \frac{1}{2}sin^2x$

[$= \frac{1}{2}.\frac{1}{2}(1 - cos2x)$, which differs from Method 1 by a constant (ie they have different constants of integration)]

Method 3 (Parts)

Integrating $cosx$:

$I = \int sinx\ cosx\ dx = sinx.sinx - \int cosx.sinx\ dx = sin^2x - I$

Hence $I = \frac{1}{2}sin^2x$

(2) $I = \int \frac{lnx}{x}\ dx$

Solution

Method 1 (substitution)

$Let\ u = lnx, so\ that\ du = \frac{1}{x}dx$

$I = \int u\, du = \frac{1}{2}u^2 = \frac{1}{2}(lnx)^2$

Method 2 (Parts)

Integrating $\frac{1}{x}$:

$I = \int \frac{lnx}{x}\, dx = lnx.lnx - \int lnx.\frac{1}{x}\, dx = (lnx)^2 - I$

Hence $I = \frac{1}{2}(lnx)^2$

(3) $I = \int \frac{x+1}{x-1} dx$

Solution

Method 1 (rearrangement)

$\mathrm{I} = \int \frac{x-1}{x-1} + \frac{2}{x-1} dx = \int 1 + \frac{2}{x-1}\, dx = x + 2\ln|x-1|$

Method 2 (substitution)

$Let\ u = x - 1, so\ that\ du = dx$

Then $I = \int \frac{u+2}{u}\, du = \int 1 + \frac{2}{u}\, du$

$= u + 2\ln|u| = x - 1 + 2\ln|x-1|$

$= x + 2\ln|x-1|$, with a different constant of integration

Note: Parts is possible, but involves integrating $\ln|x-1|$ (by Parts again), as well as a rearrangement - making it much longer than Method 1.

(4) $I = \int x(2x-1)^7 dx$

Solution

Method 1 (substitution)

$Let\ u = 2x - 1, so\ that\ du = 2dx$

$I = \int \frac{1}{2}(u+1)u^7.\frac{1}{2}du$

$= \frac{1}{4}\int u^8 + u^7 du$

$= \frac{1}{4}\left(\frac{1}{9}u^9 + \frac{1}{8}u^8\right)$

$= \frac{1}{36}(2x-1)^9 + \frac{1}{32}(2x-1)^8$

or $\frac{1}{288}(2x-1)^8[8(2x-1)+9)] = \frac{1}{288}(2x-1)^8(16x+1)$

Method 2 (Parts)

Integrating $(2x-1)^7$:

$\text{I} = x.\frac{1}{8}(2x-1)^8.\frac{1}{2} - \int \frac{1}{8}(2x-1)^8.\frac{1}{2}dx$

$= \frac{1}{16}x(2x-1)^8 - \frac{1}{16}.\frac{1}{9}(2x-1)^9.\frac{1}{2}$

$= \frac{1}{288}(2x-1)^8\{18x - (2x-1)\}$

$= \frac{1}{288}(2x-1)^8\{16x+1\}$

(5) $\int \frac{x^2}{x+1} dx$

Solution

$= \int \frac{x^2+x}{x+1} - \frac{x}{x+1} dx$

$= \int x - \frac{x+1}{x+1} + \frac{1}{x+1} dx$

$= \frac{1}{2}x^2 - x + \ln|x+1|$

(6) $I = \int \frac{x}{(1+x)^{\frac{1}{2}}} dx$

Solution

Let $u = 1 + x$, so that $du = dx$

$I = \int \frac{u-1}{u^{\frac{1}{2}}} du$

$= \int u^{\frac{1}{2}} - u^{-\frac{1}{2}} du$

$= \frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + c = \frac{2}{3}(1+x)^{\frac{3}{2}} + 2(1+x)^{\frac{1}{2}}$

(7) $I = \int x(x+1)^{\frac{3}{2}} dx$

Solution

Let $u = x + 1$, so that $du = dx$

$I = \int (u-1)u^{\frac{3}{2}} du$

$= \int u^{\frac{5}{2}} - u^{\frac{3}{2}} du$

$= \frac{2}{7}u^{\frac{7}{2}} - \frac{2}{5}u^{\frac{5}{2}}$

$= \frac{2}{7}(x+1)^{\frac{7}{2}} - \frac{2}{5}(x+1)^{\frac{5}{2}}$

$= \frac{2}{35}(x+1)^{\frac{5}{2}}(5(x+1)-14)$

$= \frac{2}{35}(x+1)^{\frac{5}{2}}(5x-9)$

[or by Parts]

(8) $\int \frac{1-x}{x+1}\,dx$

Solution

$= \int \frac{-x-1}{x+1} + \frac{2}{x+1}\ dx = \int -1 + \frac{2}{x+1}\,dx$

$= -x + 2\ln|x+1|$

(9) $\int x\sqrt{1-x}\,dx$

Solution

Let $u = 1-x$, so that $du = -dx$

Then $I = \int (1-u)u^{\frac{1}{2}}\,(-du)$

$= \int -u^{\frac{1}{2}} + u^{\frac{3}{2}}\,dx$

$= -\frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + \frac{u^{\frac{5}{2}}}{\left(\frac{5}{2}\right)}$

$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}$

(10) $\int tanx\ dx$

Solution

$= \int \frac{sinx}{cosx}\ dx$

Let $u = cosx$, so that $du = -sinx\ dx$

Then $I = \int -\frac{1}{u}\ du = -\ln|u| = -\ln|cosx| = \ln|secx|$

(11) $\int lnx\ dx$

Solution

$= \int (1) lnx\ dx = [by\ Parts]\ xlnx - \int x \left(\frac{1}{x}\right) dx$

$= xlnx - x$

(12) $\int x\sqrt{x^2 - 1}\ dx$

Solution

As $\int x\ dx = \frac{1}{2}x^2$, and the rest of the integrand is a simple function of x^2, let $u = x^2$, so that $du = 2x\ dx$, and

$I = \frac{1}{2}\int (u-1)^{\frac{1}{2}}\ du = \frac{1}{2}\ \frac{(u-1)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}$

$= \frac{1}{3}(u-1)^{\frac{3}{2}} = \frac{1}{3}(x^2 - 1)^{\frac{3}{2}}$

(13) $\int x^7 lnx\, dx$

Solution

By Parts: integrating x^7 and differentiating lnx,

$I = \frac{1}{8}x^8 lnx - \int \frac{1}{8}x^8\left(\frac{1}{x}\right)\, dx = \frac{1}{8}x^8 lnx - \frac{1}{8}\int x^7\, dx$

$= \frac{1}{8}x^8 lnx - \frac{1}{8}\left(\frac{x^8}{8}\right) = \frac{1}{8}x^8 lnx - \frac{x^8}{64}$ or $\frac{x^8}{64}(8lnx - 1)$

(14) $\int sinx\, e^x\, dx$

Solution

By Parts, integrating e^x and differentiating $sinx$:

$I = e^x sinx - \int e^x cosx\, dx$

$= e^x sinx - \{e^x cosx - \int e^x(-sinx)dx\}$

(integrating e^x and differentiating $cosx$)

$e^x sinx - e^x cosx - I$

Hence $2I = e^x(sinx - cosx)$ and $I = \frac{1}{2}e^x(sinx - cosx)$

[Note that, for the 2nd application of Parts, integrating $cosx$ and differentiating e^x will take us back to where we started:

$e^x sinx - \{sinxe^x - \int sinxe^x dx\} = I$

(15) $\int cosx\sqrt{sinx}\, dx$

Solution

Let $u = sinx$, so that $du = cosx\, dx$, and $I = \int u^{\frac{1}{2}}\,du = \frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} = \frac{2}{3}(sinx)^{\frac{3}{2}}$

(16) $\int sin^2x\;dx$

Solution

Method 1

$I = \int \frac{1}{2}(1 - cos2x)dx = \frac{1}{2}x - \frac{1}{4}sin2x$

Method 2

By Parts,

$I = \int sinxsinx\;dx = (-cosx)sinx - \int(-cosx)cosx\;dx$ (*)

$= -\frac{1}{2}sin2x + \int 1 - sin^2x\;\;dx$

$\Rightarrow 2I = -\frac{1}{2}sin2x + x$

$\Rightarrow I = \frac{1}{2}x - \frac{1}{4}sin2x$

Note: Attempting to apply Parts a 2nd time from (*):

$I = -\frac{1}{2}sin2x + \int cosxcosx\;dx$

$= -\frac{1}{2}sin2x + sinxcosx - \int sinx(-sinx)dx = I$