

Integration Theory (5 pages; 28/9/18)

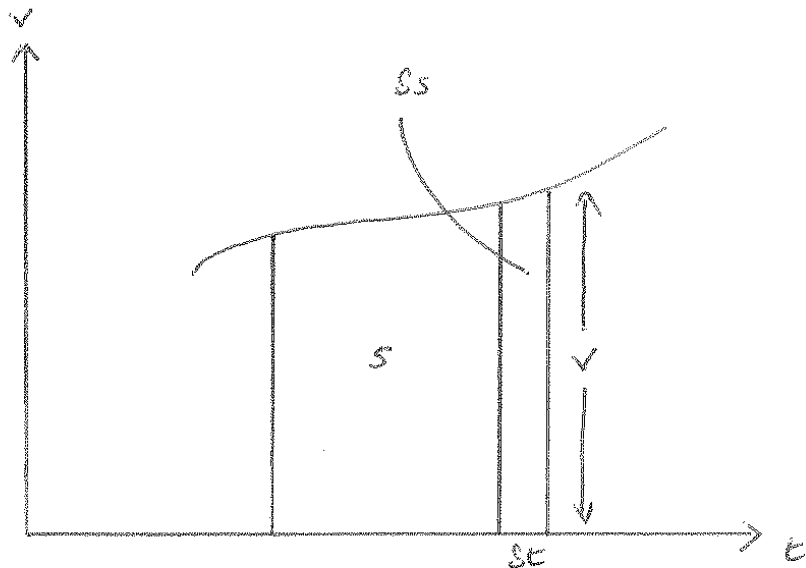
(A) The two interpretations of integration

(B) Indefinite integration

$$(C) \int \frac{1}{x} dx = \ln|x|$$

(A) The two interpretations of integration

Integration can be interpreted as either the area under a curve, or as the opposite of differentiation. To show how these two interpretations can be reconciled, refer to the diagram below.



v & s can be interpreted as speed and displacement, but the argument holds for other situations. s is defined to be the area under the curve of v , and, by the first definition of integration,

$$s = \int v dt \quad (A)$$

(to work out a specific area, limits would obviously be needed).

We want to show that integration is also the opposite of differentiation. This will be the case if $\frac{ds}{dt} = v$

From the diagram, $\frac{ds}{dt}$ is the rate at which the area increases, and is the limit as $\delta t \rightarrow 0$ of $\frac{\delta s}{\delta t}$, which equals v , since $\delta s \rightarrow v\delta t$ as $\delta t \rightarrow 0$. Thus we have shown that $\frac{ds}{dt} = v$.

In the case where v & s are speed and displacement, this works because speed is the rate of change of displacement, and displacement = speed \times time if the speed is constant (so that the displacement is the area under a horizontal line), and the natural extension of this is for the displacement to be the area under the speed-time graph in the case of a varying speed.

(B) Indefinite integration

In the definite integral $\int_{t_1}^{t_2} v(t)dt$, t is appearing as a parameter (which ranges from t_1 to t_2). It can just as easily be written as

$$\int_{t_1}^{t_2} v(x)dx$$

If t_2 is now considered to be a variable value of t , so that the definite integral represents the area under the curve as a function of t_2 , then, writing t instead of t_2 : $\int_{t_1}^t v(x)dx = s(t) - s(t_1)$

(where $v(x)$ is the derivative of $s(x)$; eg speed and displacement, respectively).

The integral is now a function of t (whereas the definite integral

$$\int_{t_1}^{t_2} v(t)dt \text{ was a fixed value}).$$

It is termed an 'indefinite' integral and, by convention, the following notation is adopted: $\int v(t)dt = s(t) + C$

C in effect equals $-s(t_1)$ and is a constant ; ie not changing with t (C is the 'constant of integration'). It can take any value (including positive values, since $s(t_1)$ can generally be made to be negative).

Note that t has been reintroduced on the left hand side, as it can no longer be confused with the upper limit of integration. This notation is slightly unsatisfactory, since the t on the left hand side is a parameter over which the integration is being carried out, whereas the t on the right hand side is the upper limit of the integration. However, the t on the left hand side does serve to indicate that the integral is to be a function of t .

(C) Given that $\int \frac{1}{x} dx = \ln x$ for $x > 0$, it can be shown that

$$\int \frac{1}{x} dx = \ln|x| \text{ for all } x \neq 0$$

Method 1

If $\int \frac{1}{x} dx = \ln x$ for $x > 0$, then $\frac{d}{dx}(\ln x) = \frac{1}{x}$ for $x > 0$

For the case where $x < 0$:

Let $y = -x$, so that $\frac{d}{dy}(\ln y) = \frac{1}{y}$, as $y > 0$

[To convert back to x s:]

Then, as $\frac{d}{dy}(\ln y) = \frac{d}{dx}(\ln y) \cdot \frac{dx}{dy}$,

it follows that $\frac{d}{dx}(\ln y) \cdot \frac{dx}{dy} = \frac{1}{(-x)}$

giving $\frac{d}{dx}(\ln[-x])(-1) = \frac{1}{(-x)}$

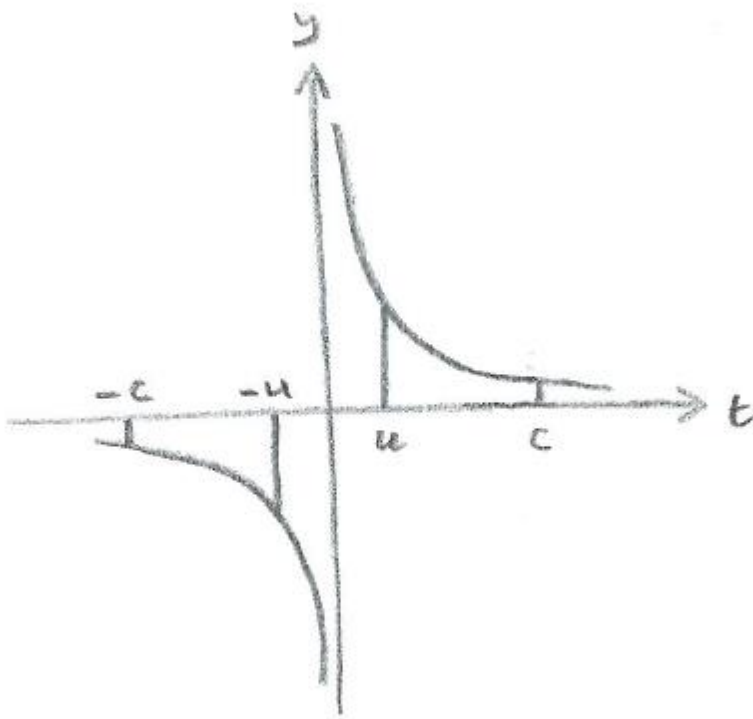
and so $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ for $x < 0$ (*)

and therefore $\int \frac{1}{x} dx = \ln|x|$ for $x < 0$, as well as $x > 0$

[Note that the function $y = \ln|x|$ for $x < 0$ is the reflection in the y -axis of $y = \ln x$ (for $x > 0$), and therefore has a negative gradient, which agrees with (*).]

Method 2

Referring to the diagram below, where $u = -x > 0$ & $c > 0$,



$$\int_{-c}^x \frac{1}{t} dt = \int_{-c}^{-u} \frac{1}{t} dt$$

= - (positive) area between graph and t -axis on LHS

= - (positive) area between graph and t -axis on RHS

$$= - \int_u^c \frac{1}{t} dt = \int_c^u \frac{1}{t} dt = \ln u - \ln c$$

As $\int \frac{1}{x} dx$ only differs from $\int_{-c}^x \frac{1}{t} dt$ by an arbitrary constant, it follows that, when $x < 0$, $\int \frac{1}{x} dx = \ln u + C = \ln|-x| + C$, as required.