Integration – Q4: Volume of Revolution / Surface Area (21/11/23)

The region between the parabola $y^2 = 4x$, the x-axis and the line $x = 1$ is rotated about the x-axis through 360°.

(i) Find the exact volume generated:

(a) by integrating with respect to x

(b) by integrating with respect to the parameter t, where $x = t^2$ and $y = 2t$

(ii) Use the mean value of the function to carry out a rough check on your answer in (i).

(iii) Find the curved surface area associated with the volume generated in (i):

(a) by integrating with respect to x

(b) by integrating with respect to y

(c) by integrating with respect to t

Solution

(i)(a) Volume $= \pi \int_0^1 y^2 dx = \pi \int_0^1 4x\, dx = \pi[2x^2]_0^1$

$= 2\pi(1-0) = 2\pi$

(b) $x = 0 \Rightarrow t = 0; x = 1 \Rightarrow t = 1$

Volume $= \pi \int_0^1 y^2 \frac{dx}{dt} dt = \pi \int_0^1 (2t)^2(2t)dt = 8\pi \int_0^1 t^3 dt$

$= 8\pi \left[\frac{1}{4}t^4\right]_0^1 = 2\pi(1-0) = 2\pi$

(ii) Mean value $= \frac{1}{1-0}\int_0^1 \sqrt{4x}\, dx = 2\int_0^1 x^{\frac{1}{2}}\, dx$

$= 2\left[\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1 = \frac{4}{3}$

Approximate volume is that of a cylinder of radius $\frac{4}{3}$ and length 1;

ie $\pi\left(\frac{4}{3}\right)^2(1) = \frac{16}{9}\pi$, which is reasonably close to 2π.

(iii)(a) Curved SA $= \int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$y = \sqrt{4x}$ and $\frac{dy}{dx} = 2(\frac{1}{2})x^{-\frac{1}{2}}$

so that $SA = 4\pi \int_0^1 x^{\frac{1}{2}}\sqrt{1+\frac{1}{x}}dx = 4\pi\int_0^1 \sqrt{x+1}\, dx$

$= 4\pi\left[\frac{(x+1)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1 = \frac{8\pi}{3}(2\sqrt{2}-1)$

(b) Curved SA $= \int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$= \int_{x=0}^{x=1} 2\pi y\sqrt{dx^2+dy^2}$ (informally)

$= \int_0^2 2\pi y\sqrt{\left(\frac{dx}{dy}\right)^2+1}\,dy$

$y^2 = 4x$, so that $\frac{dx}{dy} = \frac{1}{4}(2y)$

and SA $= 2\pi\int_0^2 y\sqrt{\frac{y^2}{4}+1}\,dy$

Then, as $\frac{d}{dy}\left(\frac{y^2}{4}\right) = \frac{2y}{4} = \frac{y}{2}$, SA $= 4\pi\left[\frac{\left(\frac{y^2}{4}+1\right)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^2$

$= \frac{8\pi}{3}(2\sqrt{2}-1)$

(c) Curved SA $= \int_0^1 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$x = t^2$ and $y = 2t$, so that $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 2$

and SA $= \int_0^1 2\pi(2t)\sqrt{4t^2+4}\,dt$

$= 4\pi\int_0^1 2t\sqrt{t^2+1}\,dt$

$= 4\pi\left[\frac{(t^2+1)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1$

$= \frac{8\pi}{3}(2\sqrt{2}-1)$