

Integration Q1: Improper Integrals (21/11/23)

Find the values of the following integrals, or show that they are not defined.

(i) $\int_{-2}^{\infty} \frac{1}{x^2} dx$

(ii) $\int_{-\infty}^{-\frac{1}{2}} e^{2x} dx$

(iii) $\int_0^1 x^{-\frac{2}{3}} dx$

Solution

$$\begin{aligned}
 \text{(i)} \quad \int_{-2}^{\infty} \frac{1}{x^2} dx &= \lim_{a \rightarrow 0^-} \int_{-2}^a \frac{1}{x^2} dx + \lim_{\substack{b \rightarrow 0^+ \\ c \rightarrow \infty}} \int_b^c \frac{1}{x^2} dx \\
 &= \lim_{a \rightarrow 0^-} \left[-\frac{1}{x} \right]_{-2}^a + \lim_{\substack{b \rightarrow 0^+ \\ c \rightarrow \infty}} \left[-\frac{1}{x} \right]_b^c
 \end{aligned}$$

which is undefined, as $\frac{1}{a}$ is undefined as $a \rightarrow 0^-$

(and similarly for b)

$$\begin{aligned}
 \text{(ii)} \quad \int_{-\infty}^{-\frac{1}{2}} e^{2x} dx &= \lim_{a \rightarrow -\infty} \int_a^{-\frac{1}{2}} e^{2x} dx \\
 &= \lim_{a \rightarrow -\infty} \left[\frac{1}{2} e^{2x} \right]_a^{-\frac{1}{2}} \\
 &= \lim_{a \rightarrow -\infty} \frac{1}{2} (e^{-1} - e^{2a}) \\
 &= \frac{1}{2e}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad \int_0^1 x^{-\frac{2}{3}} dx &= \lim_{a \rightarrow 0^+} \int_a^1 x^{-\frac{2}{3}} dx \\
 &= \lim_{a \rightarrow 0^+} \left[3x^{\frac{1}{3}} \right]_a^1 = \lim_{a \rightarrow 0^+} (3 - 3a^{\frac{1}{3}}) \\
 &= 3
 \end{aligned}$$