Integration Methods (13 pages; 6/2/18)

Contents

Also, see "Integration Exercises (Parts 1-3)".

(Note: The constant of integration has been omitted throughout.)

Introduction

The first consideration with any integration should probably be whether a rearrangement is useful, before embarking on anything too complicated. Certain integrals can of course just be found from the formulae booklet - including the use of the differentiation table (eg for $\int secx\ tanx\ dx$) Aside from this, substitution is usually the safest method. Integration by parts has various drawbacks, as will be seen.

(A) Substitution

(A1) If an integral is of the form $\int f'(x)g\big(f(x)\big)dx$

where $\int g(u)du$ can be found, then the substitution $u = f(x)$ will work,

as $du = f'(x)dx$, leaving $\int g(u)du$

If the integrand is a product of two expressions, try integrating one of them: thus $f'(x)$ integrates to $f(x)$.

Note: When the integrand is of the form $\frac{f'(x)}{f(x)}$ (ie with $g(x) = \frac{1}{x}$) , the method is sometimes described as 'by inspection', as the answer is simply $\ln(f(x))$, as can be verified by differentiation.

Example: $I = \int \tan x \ dx \ = \ \int \frac{\sin x}{\cos x} \ dx$

Integrating $sinx$ to give $-cosx$ reveals that the substitution $u = cosx$ will work:

$du = -sinx \, dx$, so that

$I = -\int \frac{1}{u} du = -lnu = -\ln(cosx) = \ln(secx)$

Example: $I = \int sinxcos^2x \, dx$

Again, $u = cosx$ works, giving $I = -\int u^2 \ du$ etc

Example: $\int \frac{e^x}{e^{2x}+1} dx$

Here $\int e^x dx = e^x$, and we can integrate $\int \frac{1}{u^2+1} \ du$

(see (C) Standard integrals)

(A2) **Example**: $I = \int x(1+x)^{\frac{1}{2}} \, dx$

Note that $(1+x)^{\frac{1}{2}}$ can only be expanded as an infinite Binomial series. However, let $u = 1 + x$, giving $I = \int (u-1)u^{\frac{1}{2}} \, du$, and the integrand can now be expanded.

(A3) 'Tidying-up' substitutions

Example: $I = \int \frac{\sin(\sqrt{x})}{\sqrt{x}} dx$

Let $u = \sqrt{x}$, $so\ that\ du = \frac{1}{2}x^{-\frac{1}{2}}\,dx$

[These tidying -up substitutions aren't guaranteed to work.]

Then $I = 2\int sinu\ du$ etc

(A4) **Example:** $I = \int \sqrt{1 - x^2}dx$

Let $x = sin\theta$, so that $dx = cos\theta\ d\theta$

Then $I = \int cos\theta cos\theta d\theta = \frac{1}{2}\int 1 + cos2\theta\ d\theta$

$= \frac{1}{2}\theta + \frac{1}{4}sin2\theta = \frac{1}{2}\theta + \frac{1}{2}sin\theta cos\theta$

$= \frac{1}{2}arcsinx + \frac{1}{2}x\sqrt{1 - x^2}$

(as can be confirmed by differentiation)

(A5) **Example**: $I = \int \frac{1}{x\sqrt{1-x^2}}dx$

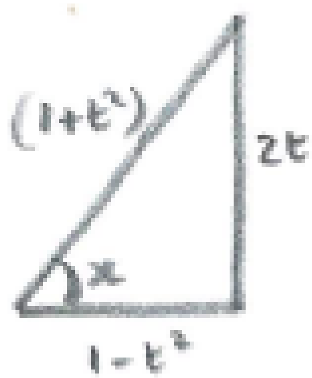
Let $u = 1/x$ so that $du = -1/x^2dx$ and $dx = -x^2du$,

so that $I = -\int \frac{ux^2}{\sqrt{1-\frac{1}{u^2}}}\ du = -\int \frac{u^2x^2}{\sqrt{u^2-1}}\ du$

$= -\int \frac{1}{\sqrt{u^2-1}}\ du = -arcoshu = -arcosh(\ 1/x)$

[This is a one-off substitution; just applicable to integrands with a factor of $1/x$]

(A6) Substitution of $t = \tan\left(\frac{x}{2}\right)$, so that



$tanx = \frac{2t}{1-t^2}$

Referring to the right-angled triangle shown,

the hypotenuse = $\sqrt{(1-t^2)^2+4t^2}$

= $\sqrt{1+2t^2+t^4}$ = 1+t^2 (conveniently)

$\frac{dt}{dx}$ = $sec^2\left(\frac{x}{2}\right).\frac{1}{2}$, so that $\frac{dx}{dt} = \frac{2}{sec^2\left(\frac{x}{2}\right)} = \frac{2}{1+t^2}$

Example: $\int secx\ dx = \int \frac{1+t^2}{1-t^2}.\frac{2}{1+t^2}$ dt = $2\int \frac{1}{1-t^2}\ dt$

= $\int \frac{1}{1-t} + \frac{1}{1+t}\ dt$ = - ln|1-t| + ln|1+t| = ln|$\frac{1+t}{1-t}$| = ln |$\frac{1+2t+t^2}{1-t^2}$|

= ln|$\frac{1+t^2}{1-t^2}+\frac{2t}{1-t^2}$| = ln|$secx + tanx$| |

(This substitution can sometimes be used to solve trig. equations as well.)

(A7) **Example**: $\int \sqrt{x^2+2x+5}\,dx = \int \sqrt{(x+1)^2+2^2}\,dx$

Let $x+1 = 2sinhy$, so that $dx = 2coshy\ dy$

and $I = 2\int \sqrt{sinh^2y+1}\ (2coshy)\ dy$

$= 4\int cosh^2y\ dy = 2\int 1+\cosh(2y)\ dy$

$= 2y+\sinh(2y)$

$= 2arsinh\left(\frac{x+1}{2}\right) + 2sinhycoshy$

$= 2arsinh\left(\frac{x+1}{2}\right) + (x+1)\sqrt{sinh^2y+1}$

$= 2arsinh\left(\frac{x+1}{2}\right) + (x+1)\sqrt{\left(\frac{x+1}{2}\right)^2+1}$

Example: $\int\sqrt{x^2+8x+7}\,dx = \int\sqrt{(x+4)^2-3^2}\,dx$

Let $x+4 = 3coshy$, so that $dx = 3sinhy\ dy$

and $I = 3\int\sqrt{cosh^2y-1}\,(3sinhy)dy$

$= 9\int sinh^2y\ dy$

$= \frac{9}{2}\int \cosh(2y) - 1)\ dy$

(and then similarly to the previous example)

Note: Alternative substitutions are $sec\theta$ in place of $coshy$, and $tan\theta$ in place of $sinhy$.

(B) Parts (incl. Reduction Formulae)

This is an obvious option whenever the integrand is a product of two expressions (or can be rearranged into this form). However, substitution will also a possibility, and may well be preferable.

The Parts formulae is derived from the product rule for differentiation:

$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$, where u & v are functions of x

Integrating both sides then gives

$uv = \int\frac{du}{dx}v\ dx + \int u\frac{dv}{dx}\ dx$

Then $\int\frac{du}{dx}v\ dx = uv - \int u\frac{dv}{dx}\ dx$

However, there is no need to write out all these symbols: note that one of the expressions in the product ($\frac{du}{dx}$) is being integrated (to u) and that the integrated expression appears in both the terms on the RHS. The expression that is to be differentiated (v) stays the same for the 1st term on the RHS, and is differentiated (to $\frac{dv}{dx}$) only in the 2nd term. In order not to get confused, the two expressions in the original integral could be labelled with an I and a D.

We may need to experiment: both to decide which term needs to be integrated, and whether the integral on the RHS is an improvement on the original integral. Sometimes we will have to apply Parts twice.

(B1) Standard situation

Example $I = \int x\, sinx\, dx$

In order to obtain a simpler integral on the RHS, we generally want to reduce the power of a term such as x^n (but see below for an exception). So x^n should generally be differentiated.
$sinkx, coskx$ & e^{kx} can be integrated or differentiated.

Here $I = x(-cosx) - \int (1)(-cosx)dx$ etc

Example $I = \int x(3x+1)^3 dx$

Integrating $(3x+1)^3$ and differentiating x,

$I = x\left(\frac{1}{4}\right)(3x+1)^4\left(\frac{1}{3}\right) - \int (1)\left(\frac{1}{4}\right)(3x+1)^4\left(\frac{1}{3}\right)dx$

$= \frac{1}{12}x(3x+1)^4 - \frac{1}{12}\int (3x+1)^4 dx$

$= \frac{1}{12}x(3x+1)^4 - \frac{1}{12}\left(\frac{1}{5}\right)(3x+1)^5\left(\frac{1}{3}\right)$

$= \frac{(3x+1)^4}{180}\{15x - (3x+1)\} = \frac{(3x+1)^4(12x-1)}{180}$

(B2) **Example**: $I = \int sinxcosx\ dx$

Integrating $cosx$ and differentiating $sinx$ (to avoid unnecessary minus signs):

$I = \int sinx\ cosx\ dx = sinx.sinx - \int cosx.sinx\ dx = sin^2x - I$

Hence $I = \frac{1}{2}sin^2x$

Example: $\int \frac{lnx}{x}\ dx$

Integrating $\frac{1}{x}$:

$I = \int \frac{lnx}{x}\ dx = lnx.lnx - \int lnx.\frac{1}{x}\ dx = (lnx)^2 - I$

Hence $I = \frac{1}{2}(lnx)^2$

(B3) Taking one of the terms of the product to be 1.

Example: $\int \ln x\ d\mathrm{x}$

write as $\int 1 . \ln x\ d\mathrm{x}$

Integrating 1,

$I = xlnx - \int x(1/x)dx = xlnx - x$

(B4) Problem case

Example: $\int \frac{1}{xlnx}\ dx$

Differentiating $\frac{1}{lnx}$:

$I = lnx\,.\frac{1}{lnx} - \int lnx\,.\ (-1)(lnx)^{-2}\left(\frac{1}{x}\right)dx$

$= 1 + I$?!

The apparent contradiction is due to the constant of integration.

Thus Parts can't be used to find this integral.

(B5) **Example:** $I = \int sinx\,.e^x\,dx$

Integrating e^x, for example,

$I = sinxe^x - \int cosxe^x dx$

Then integrating e^x again, to apply Parts for a 2nd time:

$I = sinxe^x - \{cosxe^x - \int(-sinx)e^x\,dx\}$

$= (sinx - cosx)e^x - I$

so that $I = \frac{1}{2}(sinx - cosx)e^x$

Note that when applying Parts for the 2nd time, if we had chosen to differentiate e^x, this would have led us round in circles, giving

$I = sinxe^x - \{sinxe^x - \int sinxe^x\,dx\} = I$

So, when applying Parts twice, the function resulting from integrating one of the components always has to be integrated again.

(B6) Definite integral

Example: $I = \int_1^e \left(\frac{lnx}{x}\right)^2 dx$

Writing the integrand as $\frac{1}{x^2}(lnx)^2$ and differentiating $(lnx)^2$:

$$I = \left[-\frac{1}{x}\ (lnx)^2\right]_1^e - \int_1^e \left(-\frac{1}{x}\right) 2lnx \left(\frac{1}{x}\right) dx$$

$$= -\left(\frac{1}{e} - 0\right) + 2\int_1^e \frac{lnx}{x^2} dx$$

Then differentiating lnx, to apply Parts again:

$$I = -\frac{1}{e} + 2\left[-\frac{1}{x} lnx\right]_1^e - 2\int_1^e \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right) dx$$

$$= -\frac{1}{e} - 2\left(\frac{1}{e} - 0\right) + 2\int_1^e \frac{1}{x^2} dx$$

$$= -\frac{3}{e} + 2\left[-\frac{1}{x}\right]_1^e = -\frac{3}{e} - 2\left(\frac{1}{e} - 1\right) = 2 - \frac{5}{e}$$

(B7) Reduction formulae

Integration by Parts can sometimes enable a recurrence relation to be set up.

Example: $I_n = \int_0^1 x^n e^{-x}\, dx$

Integrating e^{-x} and differentiating x^n gives:

$$I_n = [\ -e^{-x}\ x^n\ ]_0^1 \ - \int_0^1 -nx^{n-1}e^{-x}\, dx$$

$= \ -e^{-1} + 0 \ + nI_{n-1}$

Thus $I_n = nI_{n-1} - e^{-1}$

Then, since $I_0 = \int_0^1 e^{-x}\, dx = [\,-e^{-x}\,]_0^1 = -e^{-1} + 1 = 1 - e^{-1}$,

$I_1 = (1 - e^{-1}) - e^{-1} = 1 - 2e^{-1}$,

$I_2 = 2(1 - 2e^{-1}) - e^{-1} = 2 - 5e^{-1}$ etc

Example: $I_n = \int_0^{\pi} cos^n x \;\; dx$

Integrating by Parts (writing as $cosx \, . \, cos^{n-1}x$ and differentiating $cos^{n-1}x$)

leads to $I_n = \frac{n-1}{n} I_{n-2}$

Hence $\int_0^{\pi} cos^4 x = \frac{3}{4} I_2 = \frac{3}{4} \, . \, \frac{1}{2} \int_0^{\pi} 1 \, dx = \frac{3\pi}{8}$

Example: $I_n = \int_0^1 \frac{x^n}{\sqrt{1-x}} \, dx$

$I_n = [x^n \, . \, \frac{(1-x)^{1/2}}{-1/2} \,]_0^1 - \int_0^1 nx^{n-1} \frac{(1-x)^{1/2}}{-1/2} dx$

$= 0 + 2n \int_0^1 x^{n-1} . \, \frac{(1-x)}{(1-x)^{1/2}} \;\; dx$

(forcing the integrand into the form of I_n)

$= 2n(I_{n-1} - I_n)$

$\Rightarrow \; (1 + 2n)I_n = 2nI_{n-1}$

$\Rightarrow \; I_n = \frac{2nI_{n-1}}{2n+1}$

Notes

(i) Reduction formulae can also be derived for indefinite integrals.

(ii) Sometimes the integrand can be rearranged to give the reduction formula, without performing integration by Parts

eg $\int tan^n x dx = \int tan^{n-2} x(sec^2 x - 1)dx$

$= \frac{1}{n-1} tan^{n-1} x - \int tan^{n-2} x \, dx$ (as $sec^2 x$ is the derivative of $tanx$)

(iii) Reduction formulae are often associated with a proof by induction.

(iv) If $I_n = \cdots I_{n-2}$, the result will depend on whether n is odd or even.

Summary of drawbacks of Parts

(i) It may not be immediately clear which of the two expressions in the product needs to be integrated.

(ii) We may need to experiment to see whether any progress is being made, and we may not in fact arrive at a favourable integral (or we may go round in circles).

(iii) The process may be time-consuming if Parts has to be performed twice.

(C) Standard integrals

(C1) Using standard derivatives

(a) $\frac{d}{dx}(tanx)$ = sec^2x , so that $\int sec^2x\ dx = tanx + c$

(b) To find $\int cosech^2x\, dx$, note that $\frac{d}{dx}(tanhx) = sech^2x$ and establish that $\frac{d}{dx}(cothx) = -cosech^2x$

(C2) Inverse trigonometric functions

(a) $\int \frac{1}{\sqrt{a^2-x^2}}\ dx\ =\ sin^{-1}\left(\frac{x}{a}\right) + c \quad (x < |a|)$

Proof: Substitute $x = asinx$, or establish derivative of $sin^{-1}x$

$(b) \int \frac{1}{a^2+x^2}\, dx = \frac{1}{a} tan^{-1}\left(\frac{x}{a}\right) + c$

Suppose that $\int \frac{1}{a^2+x^2} dx = A\, arctan\left(\frac{x}{a}\right) + C$

Then $\frac{d}{dx} A\, arctan\left(\frac{x}{a}\right)$ = A . $\frac{1}{1+(\frac{x}{a})^2} . \frac{1}{a} = \frac{Aa}{a^2+x^2}$

Hence A = $\frac{1}{a}$

Example $I = \int \frac{2x}{1+x^4}\, dx$

Let $u = x^2$, so that $du = 2xdx$

and $I = \int \frac{1}{1+u^2} du = \arctan(x^2)$

(C3) Inverse hyperbolic functions

(a) $\int \frac{1}{\sqrt{a^2+x^2}}\, dx = sinh^{-1}\left(\frac{x}{a}\right)$ + c

(b) $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = cosh^{-1}\left(\frac{x}{a}\right)$ + c (x>a)

Note: As for $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = sin^{-1}\left(\frac{x}{a}\right)$, A=1

(D) Rearrangement

(D1) **Example** $\int \frac{1+x}{x-1}\, dx = \int \frac{x-1}{x-1}\, dx + \int \frac{2}{x-1}\, dx$ etc

(D2) Trig. relations

Example $\int tan^2x$ dx $= \int sec^2x - 1\, dx = tanx - x + c$

(D3) $\int \frac{1}{x^2-a^2}\, dx = \int \frac{1}{(x-a)(x+a)}\, dx$ etc (Partial Fractions)

(D4) Using substitution method

Example $\int sechxtanhx\, dx = \int \frac{sinhx}{cosh^2x} dx$

and then let $u = coshx$

(D5) Completing the square

Example: $\int \frac{1}{x^2+6x+13}\, dx = \int \frac{1}{(x+3)^2+4}\, dx$;

then see (C) Standard Integrals

(D6) Using the definition of a hyperbolic function

Example: $\int sechxdx = \int \frac{2}{e^x+e^{-x}} dx = 2\int \frac{e^x}{e^{2x}+1} dx$

Then let $u = e^x$, to give $2\int \frac{1}{u^2+1} du$

and see (C) Standard Integrals

(D7) 'Sum and Product Trig. Formulae'

Example

$\int \sin(mx)\cos(nx)\, dx = \frac{1}{2}\int \sin(m+n)\,x + \sin(m-n)\,x\, dx$