Inequalities (7 pages; 5/9/18)

Note: Beware of multiplying inequalities by a quantity that is (or could be) negative (eg log(0.5)).

(1) Methods for solving $\frac{x-2}{x-1} < 3$

Method 1 ('Case by case')

Case 1: $x - 1 > 0$

$\Rightarrow x - 2 < 3x - 3$

$\Rightarrow 1 < 2x \quad \Rightarrow x > \frac{1}{2}$

$x - 1 > 0$ & $x > \frac{1}{2} \Rightarrow x > 1$

Case 2: $x - 1 < 0$

$\Rightarrow x - 2 > 3x - 3$

$\Rightarrow 1 > 2x \Rightarrow x < \frac{1}{2}$

$x - 1 < 0$ & $x < \frac{1}{2} \Rightarrow x < \frac{1}{2}$

Overall solution: $x < \frac{1}{2}$ or $x > 1$

Method 2: Multiplying by the square of the denominator

$(x - 2)(x - 1) < 3(x - 1)^2$

$\Rightarrow (x - 1)\big(x - 2 - 3(x - 1)\big) < 0$

$\Rightarrow (x - 1)(-2x + 1) < 0$

$\Rightarrow (x - 1)(2x - 1) > 0$

$\Rightarrow x > 1$ & $x > \frac{1}{2}$ or $x < 1$ & $x < \frac{1}{2}$

ie $x > 1$ or $x < \frac{1}{2}$

Method 3: Rearranging into the form $\frac{f(x)}{g(x)} < 0$ etc

$\frac{x-2}{x-1} - \frac{3(x-1)}{x-1} < 0$

$\Rightarrow \frac{-2x+1}{x-1} < 0$

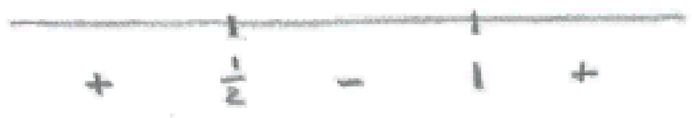
$\Rightarrow \frac{2x-1}{x-1} > 0$

critical values of x: $\frac{1}{2}$ and 1

$eg\ x = 0: LHS = 1 > 0$

$eg\ x = \frac{3}{4}: LHS = \frac{\left(\frac{1}{2}\right)}{\left(-\frac{1}{4}\right)} = -2 < 0$

$eg\ \ x = 100: LHS > 0$



Solution: $x < \frac{1}{2}$ or $x > 1$

Method 4 (Graph)

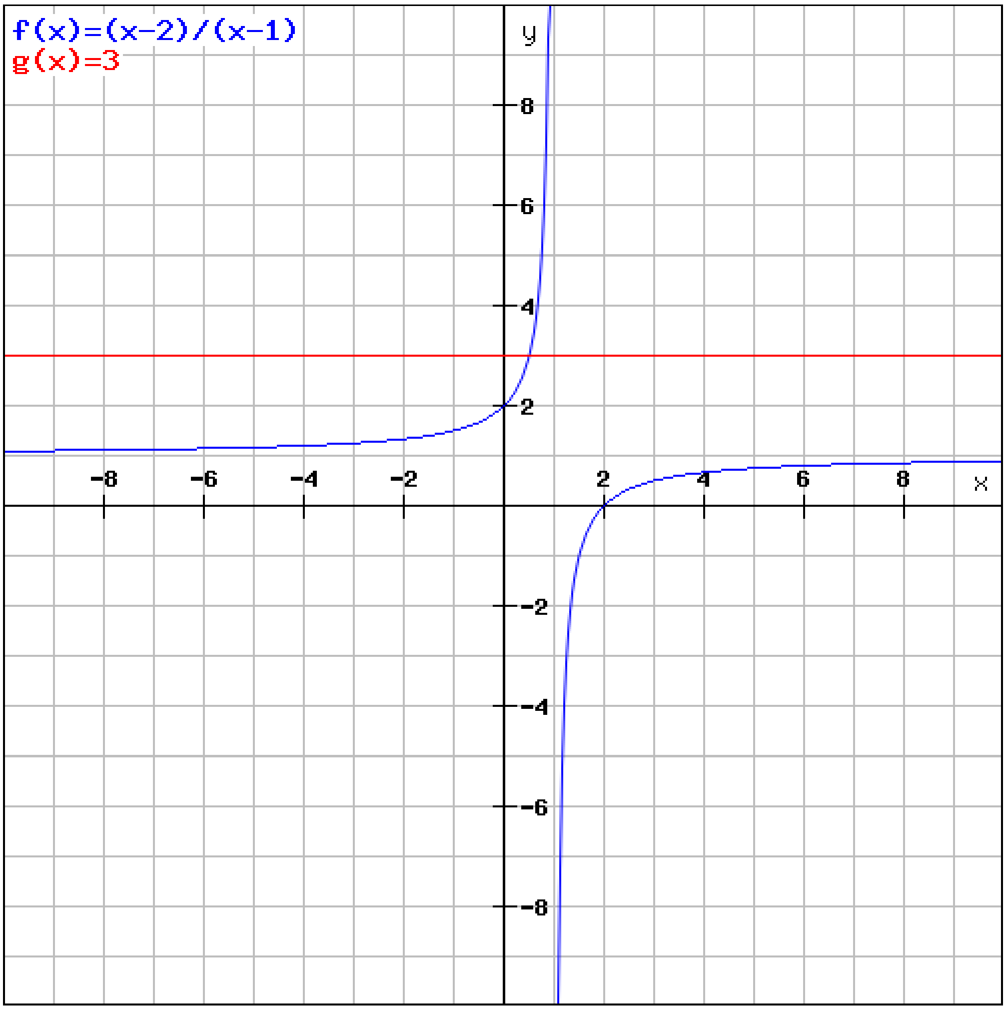
Consider the intersection of $y = \frac{x-2}{x-1}$ and $y = 3$:

$\frac{x-2}{x-1} = 3$

$\Rightarrow x - 2 = 3x - 3$

$\Rightarrow 1 = 2x \Rightarrow x = \frac{1}{2}$

$y = \frac{x-2}{x-1}$ can be sketched by noting the vertical asymptote of $x = 1$ (with $y > 0$ for $x = 1 - \delta$), and the horizontal asymptote of $y = 1$ (with $y < 1$ for eg $x = 100$)



[Noting that $\frac{x-2}{x-1} = 1 - \frac{1}{x-1}$, the graph could also be obtained from $y = \frac{1}{x}$ by the following sequence of transformations:

(i) translation of $\binom{1}{0}$, to give $y = \frac{1}{x-1}$

(ii) reflection in the x axis, to give $y = -\frac{1}{x-1}$

(iii) translation of $\binom{0}{1}$, to give $y = 1 - \frac{1}{x-1}$]

Then from the graph, $\frac{x-2}{x-1} < 3$ when $x < \frac{1}{2}$ or $x > 1$

(2) Inequalities involving moduli signs

Example A: $|x-2|>5$

Method 1

x is more than 5 away from 2, and so has to be either <-3 or >7

Method 2

$|x-2|>5 \Leftrightarrow (x-2)^2>25$ (see (3) Useful devices)

$\Leftrightarrow x^2-4x-21>0$

$\Leftrightarrow (x-7)(x+3)>0$

$\Leftrightarrow x<-3$ or $x>7$

Method 3

[Note: $|a|=a$ when $a\geq 0$ and $-a$ when $a<0$]

Case 1: $x-2\geq 0$; ie $x\geq 2$

Then $x-2>5$, so that $x>7$ and $x\geq 2$; ie $x>7$

Case 2: $x-2<0$; ie $x<2$

Then $-(x-2)>5$, so that $-x>3$ and $x<2$; ie $x<-3$

Hence $x<-3$ or $x>7$

Method 4

Draw graphs of $y=|x-2|$ and $y=5$

Example B: $2<|x+3|<7$

Method 1

$\Leftrightarrow$ distance of x from -3 is between 2 and 7

So $-10<x<-5\ or -1<x<4$

Method 2

$\Leftrightarrow 4 < (x+3)^2 < 49$

$\Leftrightarrow x^2 + 6x + 9 - 4 > 0 \; and \;\; x^2 + 6x + 9 - 49 < 0$

$\Leftrightarrow x^2 + 6x + 5 > 0 \; and \;\; x^2 + 6x - 40 < 0$

$\Leftrightarrow (x+5)(x+1) > 0 \; and \;\; (x+10)(x-4) < 0$

$\Leftrightarrow x < -5 \; or \; x > -1$ and $-10 < x < 4$

So $-10 < x < -5 \; or - 1 < x < 4$

Method 3

Case 1: $\boldsymbol{x + 3 \geq 0}$ **(ie** $\boldsymbol{x \geq -3}$**)**

Then $2 < x + 3 < 7$, so that $x > -1 \; and \; x < 4$

Hence, for case 1, $-1 < x < 4$

Case 2: $\boldsymbol{x + 3 < 0}$ **(ie** $\boldsymbol{x < -3}$**)**

Then $2 < -(x+3) < 7$, so that $-x > 5 \; and - \; x < 10$

So $x < -5 \; and \; x > -10$

Hence, for case 2, $-10 < x < -5$

Example C: $|x-2| > |x-5|$

Method 1

x has to be further from 2 than from 5

It is equidistant when $x = \frac{7}{2}$, and so has to be $> \frac{7}{2}$

Method 2

$|x-2| > |x-5| \Leftrightarrow (x-2)^2 > (x-5)^2$

$\Leftrightarrow -4x + 4 > -10x + 25$

$\Leftrightarrow 6x > 21 \Leftrightarrow x > \frac{7}{2}$

Method 3

The critical points are $x = 2$ and $x = 5$

Case 1: $x < 2$

$|x-2| > |x-5| \Leftrightarrow -(x-2) > -(x-5)$

$\Leftrightarrow 2 > 5$; ie no sol'ns

Case 2: $2 \leq x < 5$

$|x-2| > |x-5| \Leftrightarrow x-2 > -(x-5) \Leftrightarrow 2x > 7 \Leftrightarrow x > \frac{7}{2}$

So $2 \leq x < 5$ and $x > \frac{7}{2}$; ie $\frac{7}{2} < x < 5$ is a sol'n.

Case 3: $x \geq 5$

$|x-2| > |x-5| \Leftrightarrow x-2 > x-5 \Leftrightarrow -2 > -5$

So $x \geq 5$ is a sol'n.

So $x > \frac{7}{2}$

Method 4

Draw the graphs of $y = |x-2|$ and $y = |x-5|$

(3) Useful devices

(i) If a & b are ≥ 0, then $a > b \Leftrightarrow a^2 > b^2$ (as $y = x^2$ is an increasing function for $x \geq 0$).

eg $|x-1| > |x+2| \Leftrightarrow (x-1)^2 > (x+2)^2$

(ii) If an expression can be arranged into the form $(a-b)^2$, then this will be non-negative.

(iii) Consider critical values where equality holds.

(iv) $a < b \Rightarrow \frac{1}{a} + \frac{1}{b} < \frac{1}{a-\delta} + \frac{1}{b+\delta}$ $(\delta > 0)$

(v) Use of linear interpolation, to obtain lower or upper bound.

(4) Exercises

Are the following true or false?

(i) $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$

(ii) $a < b \Rightarrow a^2 < b^2$

(iii) $a < b \,\&\, c < d \Rightarrow a + c < b + d$

(iv) $a < b \,\&\, c < d \Rightarrow a - c < b - d$

Solutions

(i) Not true if $a < 0 \,\&\, b > 0$ (consider the graph of $y = 1/x$)

(ii) Not true if $a < 0 \,\&\, b < 0$ or

if $a < 0, b > 0 \,\&\, |b| < |a|$(consider the graph of $y = x^2$)

(iii) True: $a < b \Rightarrow a + c < b + c < b + d$

(iv) False: For example, $8 < 9$ and $2 < 4$, but it is not true that $8 - 2 < 9 - 4$; see diagram