Induction (4 pages; 22/8/16)

Example 1

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

Solution

$\sum_{r=1}^{1} r^3 = 1$ and $\frac{1}{4}(1)^2(1+1)^2 = 1$; thus the result is true for $n = 1$

Now assume that the result is true for $n = k$, so that

$\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$

[A common error is to write "Let $n = k$", or "Assume $n = k$", or "If $n = k$"]

[At this point it is possible to indicate the 'target' for $n = k + 1$; namely that $\sum_{r=1}^{k+1} r^3 = \frac{1}{4}(k+1)^2(k+2)^2$]

Then $\sum_{r=1}^{k+1} r^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$

$= \frac{1}{4}(k+1)^2(k^2 + 4(k+1))$

$= \frac{1}{4}(k+1)^2(k+2)^2$

$= \frac{1}{4}(k+1)^2([k+1]+1)^2$

But this is the result with k replaced by $k + 1$.

[Or, if the target has been mentioned previously, just indicate that the target has been obtained.]

So, if the result is true for $n = k$, then it is true for $n = k + 1$.

[A common error is to write: "So the result is true for $n = k$ and $n = k + 1$"]

As the result is true for $n = 1$, it is therefore true for $n = 2, 3, \ldots$ and, by the principle of [mathematical] induction, for all integer $n \geq 1$. [In some cases, it is appropriate to start at a different value for n, such as 0 or 2. This depends on what values of n the given formula is defined for.]

[Note that no credit is ever given in an exam for the highlighted wording on its own, or where the algebra is 'fudged'.]

Example 2

(1) If $u_{n+1} = 2u_n + 3$, where $u_1 = 5$, then $u_n = 2^{n+2} - 3$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$, so that

$u_k = 2^{k+2} - 3$

Then $u_{k+1} = 2u_k + 3 = 2(2^{k+2} - 3) + 3 = 2^{k+3} - 3$

$= 2^{(k+1)+2} - 3$

[Standard wording]

Example 3

(1) $7^n + 4^n + 1$ is divisible by 6

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $7^k + 4^k + 1 = 6M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$7^{k+1} + 4^{k+1} + 1 = 7(7^k + 4^k + 1) - 3(4^k) - 6$

$= 7(6M) - 6(2)(4^{k-1}) - 6$

$= 6(7M - 2(4^{k-1}) - 1)$, which is a multiple of 6 for $k \geq 1$

(the multiple is positive, as $7^{k+1} + 4^{k+1} + 1$ is positive)

[Standard wording]

Approach 2

Let $f(k) = 7^k + 4^k + 1$

Then $f(k+1) - \lambda f(k) = (7^{k+1} + 4^{k+1} + 1) - \lambda(7^k + 4^k + 1)$

[an appropriate λ will be chosen shortly]

$= 7^k(7 - \lambda) + 4^k(4 - \lambda) + 1 - \lambda$

Let $\lambda = 7$, so that $f(k+1) - 7f(k) = -3(4^k) - 6$

and $f(k+1) = 7f(k) - 6(2)(4^{k-1}) - 6$

As $f(k)$ is assumed to be a multiple of 6, and the other terms on the RHS are also multiples of 6 (for $k \geq 1$), it follows that $f(k+1)$ is a multiple of 6 (the multiple is positive, as

$7^{k+1} + 4^{k+1} + 1$ is positive).

[Standard wording]

[Approach 1 is in fact a quicker version of Approach 2, although it looks a bit more speculative. It expresses $f(k+1)$ as

$\lambda f(k) + g(k)$ ($g(k) = -3(4^k) - 6$ in this example)

and, in almost all cases, $g(k)$ will not contain any more multiples of $f(k)$, so that we have $f(k+1) - \lambda f(k) = g(k)$, as in Approach 2.

Textbooks sometimes consider $f(k+1) - f(k)$ (ie with $\lambda = 1$), but this usually involves more work (with $f(k+1) - f(k)$ having to be written in the form $(\lambda - 1)f(k) + g(k)$).