Induction - Exercises (3 pages; 22/8/16)

Prove the following results by mathematical induction.

Type A [not an official term!]

(1) The sum of the 1st n odd numbers is n^2

(2) $1 \times 4 + 2 \times 5 + 3 \times 6 + \cdots + n(n+3) = \frac{1}{3}n(n+1)(n+5)$

(3) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

(4) $2 + 4 + 6 + \cdots + 2n = n(n+1)$

(5) $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$

(6) $\sum_{r=1}^{n} r(r+2) = \frac{1}{6}n(n+1)(2n+7)$

(7) $\sum_{r=1}^{n} r(r+1)(r+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$

(8) $\sum_{r=1}^{n} 2^r = 2(2^n - 1)$

(9) $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$

(10) $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

(11) $\sum_{r=1}^{n} r(r!) = (n+1)! - 1$

Type B

(1) If $u_n = u_{n-1} + 2$, where $u_1 = 3$, then $u_n = 2n + 1$

(2) If $u_n = 3u_{n-1} + 4$, where $u_1 = 2$, then $u_n = 4(3^{n-1}) - 2$

(3) If $u_n = 3u_{n-1} - 2u_{n-2}$, where $u_1 = 1$ & $u_2 = 3$,

then $u_n = 2^n - 1$

(4) If $u_n = 5u_{n-1} - 6u_{n-2}$, where $u_0 = -1$ & $u_1 = -1$,

then $u_n = 3^n - 2^{n+1}$

(5) If $u_{n+1} = 3u_n - 2^n$, where $u_1 = 5$, then $u_n = 2^n + 3^n$

(6) If $u_{n+1} = 4n - u_n$, where $u_1 = \frac{1}{2}$,

then $u_n = 2n + \frac{1}{2}(-1)^n - 1$

(7) If $u_{n+1} = \frac{u_n}{u_n+1}$, where $u_n = 1$, suggest a formula for u_n and prove it by induction

Type C

(1) $7^{2n-1} + 3^{2n}$ is divisible by 8

(2) $2^{n+2} + 3^{2n+1}$ is divisible by 7

(3) $5^n + 12n - 1$ is divisible by 16

(4) $2^{n+1} + 9(13^n)$ is divisible by 11

(5) $13^n + 6^{n-1}$ is divisible by 7

(6) $5^{2n} + 12^{n-1}$ is divisible by 13

(7) $5^{2n+2} - 24n - 25$ is divisible by 576

(8) $2^{4n+1} + 3$ is divisible by 5

Type D (Miscellaneous)

(1) $2 + 4 + 6 + \cdots + 2n > n^2$

(2) $\sum_{r=1}^{n} r^2 > \frac{1}{3}n^3$

(3) $\frac{1}{4}n^4 < \sum_{r=1}^{n} r^3 \leq n^4$

(4) The sum of the interior angles of a convex n-sided polygon is $180(n-2)$

(5) If $A = \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$, then $A^n = \begin{pmatrix} 1-2n & -4n \\ n & 1+2n \end{pmatrix}$

(6) $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$ for $n \geq 2$

(7) (i) If $y = e^x sinx$, show that $\frac{dy}{dx} = \sqrt{2}\, e^x \sin(x + \frac{\pi}{4})$

(ii) Show that $\frac{d^n y}{dx^n} = \left(\sqrt{2}\right)^n e^x \sin(x + \frac{n\pi}{4})$