Induction – Q29 [Practice/M] (18/6/23)

$\frac{1}{4}n^4 < \sum_{r=1}^{n} r^3 \leq n^4$

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

so that $\frac{1}{4}k^4 < \sum_{r=1}^{k} r^3 \leq k^4$

Then $\sum_{r=1}^{k+1} r^3 > \frac{1}{4}k^4 + (k+1)^3$

$= \frac{1}{4}(k+1)^4 - \frac{1}{4}(4k^3 + 6k^2 + 4k + 1) + (k+1)^3$

$= \frac{1}{4}(k+1)^4 - \frac{1}{4}(6k^2 + 4k + 1) + 3k^2 + 3k + 1$

$= \frac{1}{4}(k+1)^4 + \frac{3}{2}k^2 + 2k + \frac{3}{4}$

$> \frac{1}{4}(k+1)^4$ (as $k > 0$)

Also, $\sum_{r=1}^{k+1} r^3 \leq k^4 + (k+1)^3$

$= (k+1)^4 - 4k^3 - 6k^2 - 4k - 1 + (k+1)^3$

$= (k+1)^4 - 4k^3 - 6k^2 - 4k - 1 + k^3 + 3k^2 + 3k + 1$

$= (k+1)^4 - 3k^3 - 3k^2 - k$

$< (k+1)^4$ (as $k > 0$)

$\leq (k+1)^4$

[Standard wording]