Induction – Q19 [Practice/E] (18/6/23)

$7^{2n-1} + 3^{2n}$ is divisible by 8

Solution

[Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$

Approach 1

so that $7^{2k-1} + 3^{2k} = 8M$, where $M \in \mathbb{Z}^+$

To show that the result is then true for $n = k + 1$:

$7^{2(k+1)-1} + 3^{2(k+1)} = 7^{2k+1} + 3^{2k+2}$

$= 49(7^{2k-1}) + 3^{2k+2}$

$= 49(8M - 3^{2k}) + 3^{2k+2}$

$= 8(49M) + 3^{2k}(-49 + 9)$

$= 8(49M - 5(3^{2k}))$

(the multiple is positive, as $7^{2(k+1)-1} + 3^{2(k+1)}$ is positive)

[Standard wording]

Approach 2

Note: This approach is sometimes suggested with $\lambda = 1$, but there is no guarantee that it will work for this value of λ, as can be seen in subsequent examples.

Let $f(k) = 7^{2k-1} + 3^{2k}$

Then $f(k+1) - \lambda f(k) = 7^{2k+1} + 3^{2k+2} - \lambda(7^{2k-1} + 3^{2k})$

$= 7^{2k-1}(49 - \lambda) + 3^{2k}(9 - \lambda)$

Let $\lambda = 1$, so that $f(k+1) = f(k) + 48(7^{2k-1}) + 8(3^{2k})$

As $f(k)$ is assumed to be a multiple of 8, and the other terms on the RHS are also a multiple of 8, it follows that $f(k+1)$ is a multiple of 8. [Standard wording]