

Hyperbolic Functions: Exercises - Sol'ns (9 pages; 13/5/17)

(1) (i) Prove, using exponential functions, that

(a) $\cosh^2 x - \sinh^2 x = 1$

(b) $\sinh 2x = 2 \sinh x \cosh x$

(ii) By differentiating the result from (i)(b), obtain an expression for $\cosh 2x$ in terms of $\cosh^2 x$ and $\sinh^2 x$

Solution

(i)(a) As $\cosh x = \frac{1}{2}(e^x + e^{-x})$ & $\sinh x = \frac{1}{2}(e^x - e^{-x})$,

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= (\cosh x + \sinh x)(\cosh x - \sinh x) \\ &= e^x \cdot e^{-x} = 1 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad 2 \sinh x \cosh x &= 2 \left(\frac{1}{2}\right)(e^x - e^{-x}) \left(\frac{1}{2}\right)(e^x + e^{-x}) \\ &= \frac{1}{2}(e^{2x} - e^{-2x}) = \sinh 2x \quad (\text{by difference of 2 squares}) \end{aligned}$$

(ii) Differentiating $\sinh 2x = 2 \sinh x \cosh x$ gives

$$\begin{aligned} 2 \cosh 2x &= 2 \cosh x \cosh x + 2 \sinh x \sinh x \\ \Rightarrow \cosh 2x &= \cosh^2 x + \sinh^2 x \end{aligned}$$

(2) (a) Find the formula connecting $\tanh^2 x$ & $\operatorname{sech}^2 x$?

(b) Find the formula connecting $\operatorname{coth}^2 x$ & $\operatorname{cosech}^2 x$?

Solution

From $\cosh^2 x - \sinh^2 x = 1$,

(a) divide by $\cosh^2 x$, to give $1 - \tanh^2 x = \operatorname{sech}^2 x$

(b) divide by $\sinh^2 x$, to give $\operatorname{coth}^2 x - 1 = \operatorname{cosech}^2 x$

(3) Show that $\operatorname{artanh} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$ ($|x| < 1$)

Solution

If $y = \operatorname{artanh} x$, then $\operatorname{tanh} y = x$ ($|x| < 1$)

$$\Rightarrow x = \frac{\frac{1}{2}(e^y - e^{-y})}{\frac{1}{2}(e^y + e^{-y})} = \frac{e^{2y} - 1}{e^{2y} + 1}$$

$$\Rightarrow x(e^{2y} + 1) = e^{2y} - 1$$

$$\Rightarrow e^{2y}(x - 1) = -1 - x$$

$$\Rightarrow e^{2y} = \frac{1+x}{1-x}$$

$$\Rightarrow y = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) \quad (|x| < 1)$$

(4) Differentiation

Find or prove the following:

(i) $\frac{d}{dx} \operatorname{tanh} x$

(ii) $\frac{d}{dx} \operatorname{arcosh} x = \frac{1}{\sqrt{x^2 - 1}}$

(iii) $\frac{d}{dx} \operatorname{artanh} x = \frac{1}{1-x^2}$

(iv) $\frac{d}{dx} \operatorname{sech} x$

Solutions

$$\begin{aligned} \text{(i)} \quad \frac{d}{dx} \operatorname{tanh} x &= \frac{d}{dx} \frac{\sinh x}{\cosh x} = \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{\cosh^2 x} \\ &= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x \end{aligned}$$

(ii) Let $y = \operatorname{arcosh} x$, so that $\operatorname{cosh} y = x$

Then $\frac{dx}{dy} = \sinh y$ and $\frac{dy}{dx} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$

(iii) Let $y = \operatorname{artanh} x$, so that $\tanh y = x$

and $\frac{dx}{dy} = \operatorname{sech}^2 y = 1 - \tanh^2 y = 1 - x^2$

Hence $\frac{d}{dx} \operatorname{artanh} x = \frac{1}{1-x^2}$

(iv) $\frac{d}{dx} \operatorname{sech} x = \frac{d}{dx} (\cosh x)^{-1} = (-1)(\cosh x)^{-2} \sinh x$

$= -\operatorname{sech}^2 x \cdot \sinh x$ or $-\operatorname{sech} x \cdot \tanh x$

[Although this is similar to $\frac{d}{dx} \sec x = \sec x \cdot \tan x$, Osborn's rule doesn't apply to derivatives (and, in any case, there is no $\sinh^2 x$ or similar term).]

(5) Simplify $\sinh(\cosh^{-1} 2)$

Solution

Let $\cosh^{-1} 2 = a$, so that $2 = \cosh a$

Then $\sinh a = \sqrt{\cosh^2 a - 1} = \sqrt{3}$

(6) Solve the equation $5\cosh 2x + 3\sinh x = 6$,

giving your answers in exact logarithmic form

Solution

$$5\cosh 2x + 3\sinh x = 6 \Rightarrow 5(\cosh^2 x + \sinh^2 x) + 3\sinh x - 6 = 0$$

$$\Rightarrow 5(1 + 2\sinh^2 x) + 3\sinh x - 6 = 0$$

$$\Rightarrow 10\sinh^2 x + 3\sinh x - 1 = 0$$

$$\Rightarrow (5\sinh x - 1)(2\sinh x + 1) = 0$$

$$\Rightarrow \sinh x = \frac{1}{5} \text{ or } -\frac{1}{2}$$

$$\Rightarrow x = \operatorname{arsinh}\left(\frac{1}{5}\right) \text{ or } \operatorname{arsinh}\left(-\frac{1}{2}\right)$$

$$\Rightarrow x = \ln\left(\frac{1}{5} + \sqrt{\frac{1}{25} + 1}\right) \text{ or } \ln\left(-\frac{1}{2} + \sqrt{\frac{1}{4} + 1}\right)$$

$$\text{or } \ln\left(\frac{1}{5}(1 + \sqrt{26})\right) \text{ or } \ln\left(\frac{1}{2}(\sqrt{5} - 1)\right)$$

[It is possible to substitute these values into the equation, as a check.]

$$(7) \text{ Show that } \operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1})$$

Solution

If $y = \operatorname{arcosh} x$, then $\cosh y = x$

$$\Rightarrow x = \frac{1}{2}(e^y + e^{-y})$$

$$\Rightarrow 2xe^y = e^{2y} + 1$$

$$\Rightarrow e^{2y} - 2xe^y + 1 = 0$$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

$$\Rightarrow y = \ln(x \pm \sqrt{x^2 - 1})$$

However, in order for $y = \operatorname{arcosh} x$ to be a function, the negative branch is suppressed (by restricting the domain of $\cosh x$ to non-negative values). And we can show that $\ln(x - \sqrt{x^2 - 1}) < 0$:

Method 1

Equivalently, we need to show that $x - \sqrt{x^2 - 1} < 1$; or that

$$x - 1 < \sqrt{x^2 - 1}$$

But $x - 1 = \sqrt{(x - 1)(x - 1)}$ (noting that the range of $\cosh x$, and hence the domain of $\operatorname{arcosh} x$, excludes $x < 1$)

and $\sqrt{(x - 1)(x - 1)} < \sqrt{(x - 1)(x + 1)} = \sqrt{x^2 - 1}$, as required

($y = \sqrt{x}$ is an increasing function,

so $x - 1 < x + 1 \Rightarrow \sqrt{x - 1} < \sqrt{x + 1}$)

[Alternatively, we can argue (slightly informally) that the difference between x^2 and $x^2 - 1$ (ie 1) is contracted by applying the square root function, so that $\sqrt{x^2} - \sqrt{x^2 - 1} < 1$]

Method 2

We expect the unrestricted $y = \operatorname{arcosh} x$ to be symmetric about the x -axis (as $y = \cosh x$ is symmetric about the y -axis). So we could show that $y = \ln(x \pm \sqrt{x^2 - 1})$ can also be written as

$y = \pm \ln(x + \sqrt{x^2 - 1})$, and then reject the negative branch as before.

So we want to show that $\ln(x - \sqrt{x^2 - 1}) = -\ln(x + \sqrt{x^2 - 1})$:

$$\text{RHS} = \ln\left(\frac{1}{x + \sqrt{x^2 - 1}}\right) = \ln\left(\frac{x - \sqrt{x^2 - 1}}{x^2 - (x^2 - 1)}\right) = \ln(x - \sqrt{x^2 - 1}), \text{ as}$$

required

(8) If $x = \sinh u$, write $\sinh(4u)$ in terms of x

Solution

$$\begin{aligned} \sinh(4u) &= 2 \sinh(2u) \cosh(2u) \\ &= 4 \sinh u \cosh u (\cosh^2 u + \sinh^2 u) \\ &= 4x \sqrt{1 + x^2} (1 + 2x^2) \end{aligned}$$

(9) Derive an expression for $\operatorname{arsinh}(a)$ in the form $\ln b$

Solution

Let $x = \operatorname{arsinh}(a)$, so that $\sinh x = a$

$$\text{and } \frac{1}{2}(e^x - e^{-x}) = a$$

$$\text{Then } \frac{1}{2}(e^{2x} - 1) = ae^x$$

$$\text{and } e^{2x} - 2ae^x - 1 = 0,$$

so that $e^x = \frac{2a \pm \sqrt{4a^2 + 4}}{2} = a + \sqrt{a^2 + 1}$ (rejecting the negative root)

$$\text{Thus } \operatorname{arsinh}(a) = \ln(a + \sqrt{a^2 + 1})$$

(noting that $a + \sqrt{a^2 + 1} > 0$)

$$\text{Note that } \operatorname{arsinh}\left(\frac{x}{a}\right) = \ln\left(\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 + 1}\right)$$

$$= \ln\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right) = \ln(x + \sqrt{x^2 + a^2}) - \ln a$$

In the formulae booklets, $\int \frac{1}{\sqrt{a^2 + x^2}} dx$ is often given as

" $\operatorname{arsinh}\left(\frac{x}{a}\right)$ or $\ln(x + \sqrt{x^2 + a^2})$ " but, as we've just seen, these two expressions differ by a constant

(10) Given that $\operatorname{artanh} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$ and $\operatorname{arcoth} x = \frac{1}{2} \ln\left(\frac{1+x}{x-1}\right)$,

$$\text{and also that } \frac{d}{dx}(\operatorname{artanh} x) = \frac{d}{dx}(\operatorname{arcoth} x) = \frac{1}{1-x^2},$$

what is wrong with the following reasoning?

$$\int \frac{1}{1-x^2} dx = \operatorname{artanh} x + C = \operatorname{arcoth} x + C_1,$$

so that $\operatorname{artanh}x - \operatorname{arcoth}x = C_2$

$$\text{But } \operatorname{artanh}x - \operatorname{arcoth}x = \frac{1}{2} \ln \left(\frac{\left(\frac{1+x}{1-x}\right)}{\left(\frac{1+x}{x-1}\right)} \right) = \frac{1}{2} \ln \left(\frac{x-1}{1-x} \right) = \frac{1}{2} \ln(-1),$$

which isn't defined!

Solution

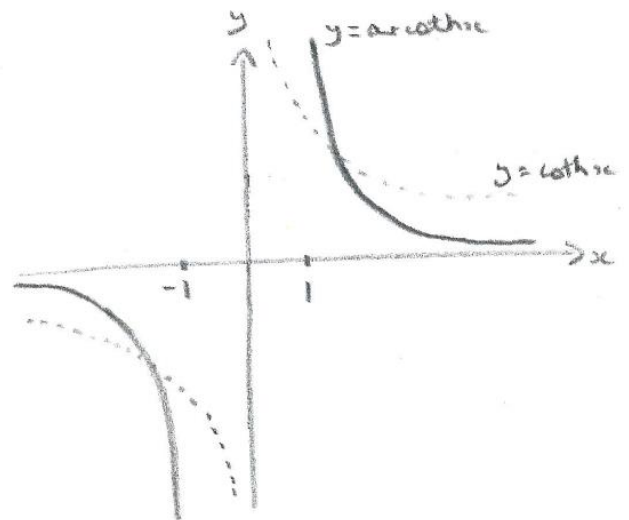
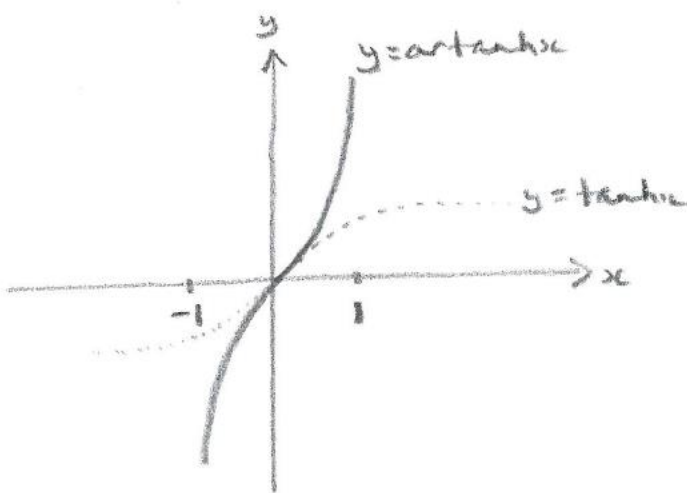
The problem is that the domains of $y = \operatorname{artanh}x$ and

$y = \operatorname{arcoth}x$ don't overlap (see graphs below). We ought to say

that $\operatorname{artanh}x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$ for $|x| < 1$ and $\operatorname{arcoth}x = \frac{1}{2} \ln \left(\frac{1+x}{x-1} \right)$

for $|x| > 1$. So it doesn't make sense to determine

$\operatorname{artanh}x - \operatorname{arcoth}x$



Note that, with $|x| < 1$, $\frac{d}{dx}(\operatorname{artanh}x) = \frac{1}{1-x^2} > 0$ for all x ; whilst

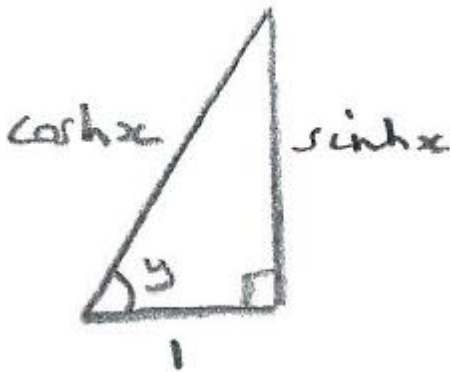
with $|x| > 1$, $\frac{d}{dx}(\operatorname{arcoth}x) = \frac{1}{1-x^2} < 0$ for all x

(11) Given that $\sinh x = \tanh y$, where $-\frac{\pi}{2} < y < \frac{\pi}{2}$, show that

(a) $\tanh x = \sin y$ (b) $x = \ln(\tanh y + \sec y)$

Solution

(a) As $\sinh x = \tanh y$, we can construct a right-angled triangle (see diagram below), where the hypotenuse is $\cosh x$, as $\sinh^2 x + 1 = \cosh^2 x$.



Then $\sin y = \frac{\sinh x}{\cosh x} = \tanh x$, as required.

Alternatively: $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\tanh y}{\sqrt{1 + \sinh^2 x}}$

(from $\sinh^2 x + 1 = \cosh^2 x$, noting that $\cosh x$ is always positive, so that we take the positive square root)

$$= \frac{\tanh y}{\sqrt{1 + \tanh^2 y}} = \frac{\tanh y}{\sqrt{\sec^2 y}} = \frac{\tanh y}{\sec y}$$

(as $\cos y > 0$ when $-\frac{\pi}{2} < y < \frac{\pi}{2}$, and hence $\sec y > 0$ also)

$$= \tanh y \cos y = \sin y$$

(b) From the right-angled triangle,

$$\tanh y + \sec y = \sinh x + \cosh x$$

$$= \frac{1}{2}(e^x - e^{-x}) + \frac{1}{2}(e^x + e^{-x}) = e^x,$$

so that $\ln(\tanh y + \operatorname{sech} y) = x$, as required.

Alternatively: $\sinh x = \tanh y \Rightarrow \frac{1}{2}(e^x - e^{-x}) = \tanh y$

$$\Rightarrow e^{2x} - 1 = 2 \tanh y e^x$$

$$\Rightarrow e^{2x} - 2 \tanh y e^x - 1 = 0$$

$$\Rightarrow e^x = \frac{2 \tanh y \pm \sqrt{4 \tanh^2 y + 4}}{2} = \tanh y \pm \operatorname{sech} y$$

$$\tanh y - \operatorname{sech} y = \frac{\sinh y - 1}{\cosh y} < 0 \text{ when } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Hence, as $e^x > 0$, it follows that $e^x = \tanh y + \operatorname{sech} y$,

and hence $x = \ln(\tanh y + \operatorname{sech} y)$

(12) What is the domain of $\operatorname{artanh}\left(\frac{x}{2}\right)$?

Solution

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{e^{2x} + 1}{e^{2x} + 1} - \frac{2}{e^{2x} + 1} = 1 - \frac{2}{e^{2x} + 1}$$

Thus $-1 < \tanh x < 1$ (as $x \rightarrow -\infty$ & ∞)

As $\operatorname{artanh} x$ is the inverse of $\tanh x$, the domain of $\operatorname{artanh} x$ is the range of $\tanh x$; ie $(-1, 1)$.

Thus the domain of $\operatorname{artanh}\left(\frac{x}{2}\right)$ satisfies $-1 < \frac{x}{2} < 1$;

ie $-2 < x < 2$