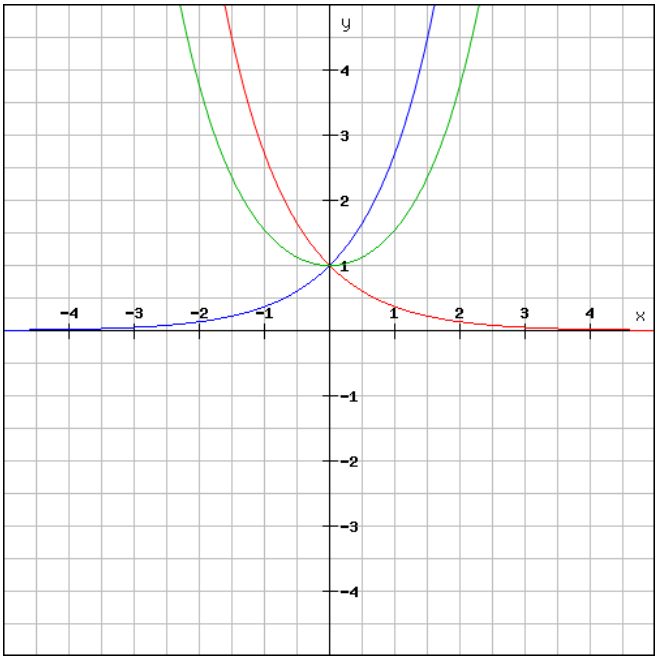
Hyperbolic Functions (10 pages; 26/7/16)

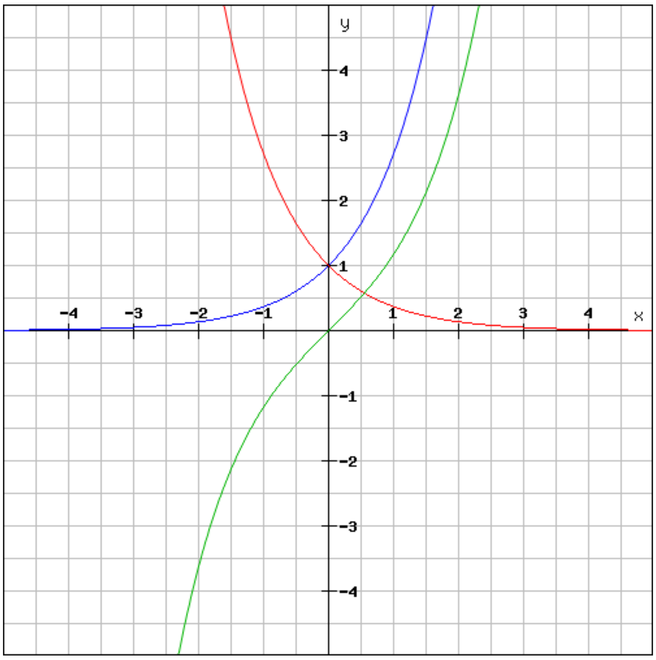
[See also Worked Exercises]

(1) By definition, $coshx = \frac{1}{2}(e^x + e^{-x})$ [green]

(compared in the diagram with $y = e^x$ [blue] & $y = e^{-x}$ [red])



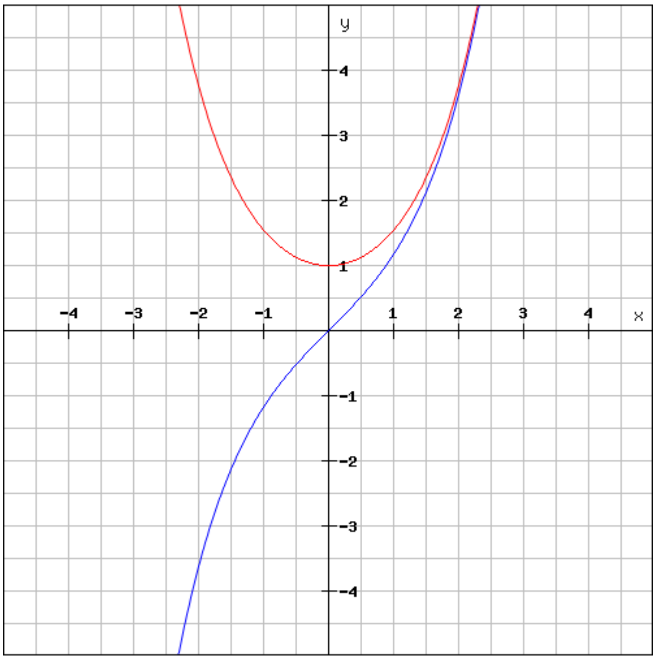
(2) $sinhx = \frac{1}{2}(e^x - e^{-x})$ [usually pronounced 'shine'; I believe Americans say 'sinch']



(3) $\cosh(-x) = coshx$ (an 'even' function)

$\sinh(-x) = -sinhx$ (an 'odd' function)

e^x can be written as $coshx + sinhx$



$y = coshx\ [red]$ & $y = sinhx\ [blue]$

(4) $tanhx$ [pronounced either as 'tanch' or 'than' (as in Thanet)]

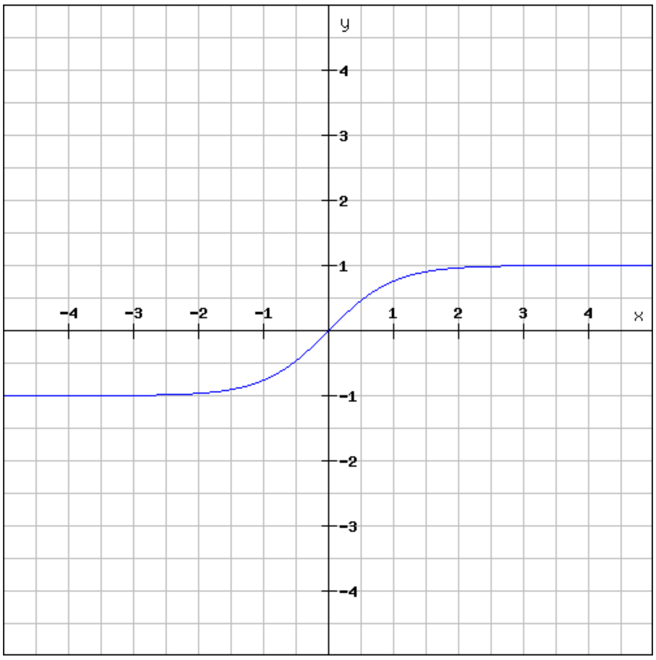
$tanhx$ is defined as $\frac{sinhx}{coshx}$

As $x \to \infty, sinhx \to coshx$

and so $tanhx = \frac{sinhx}{coshx} \to 1$

As $x \to -\infty, sinhx \to -coshx$

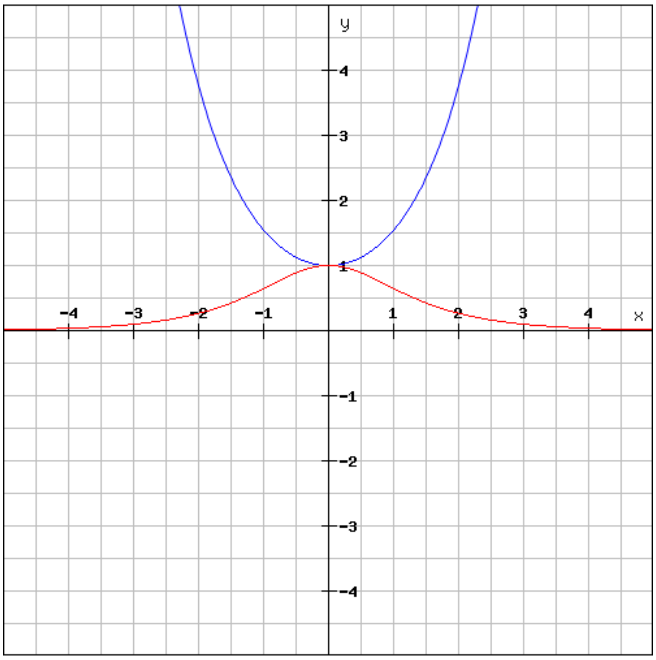
and so $tanhx \to -1$



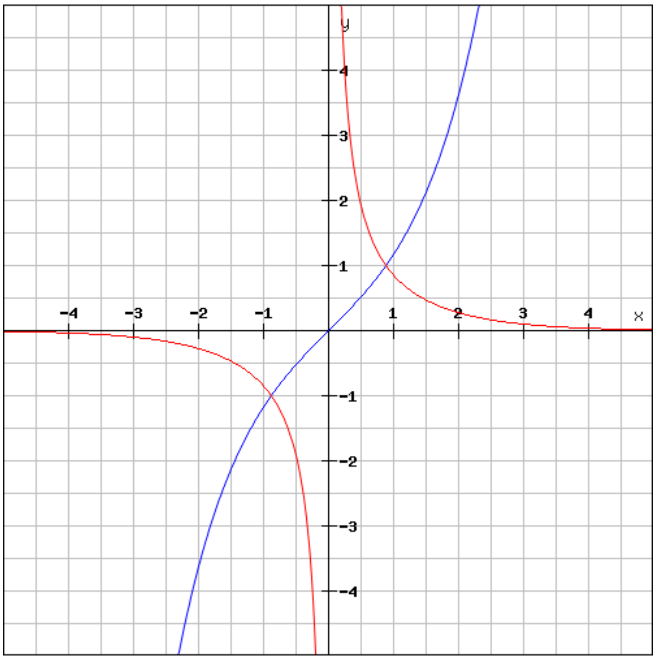
$y = tanhx$

(5) Reciprocal Functions

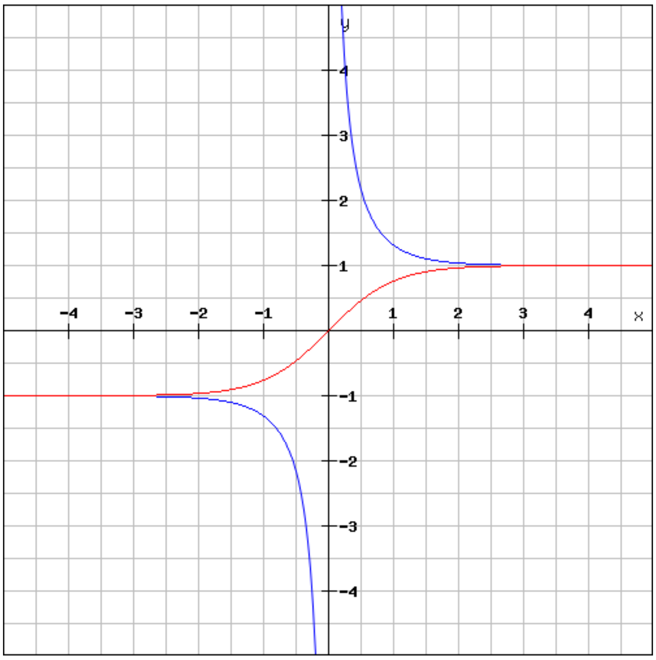
$sechx = \frac{1}{coshx}$ [pronounced 'sess', as in session, or 'sheck'] (red graph below)



$cosechx = \frac{1}{sinhx}$ (red graph below)



$cothx = \frac{1}{tanhx}$ (blue graph below)



(6) $cosh^2x - sinh^2x = 1$

(See Exercises for proof.)

Note that $cosh^2x = sinh^2x + 1$; reflecting the fact that $coshx \geq 1$

(7) $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$ is the Cartesian equation of a hyperbola

A parametric form for the right-hand branch of this hyperbola is

$x = acosht,\ y = bsinht$

(the left-hand branch is obtained from $x = -acosht,\ y = bsinht$)

[$x = \sec(t)$, $y = \tan(t)$ is the more usual form, which covers both branches]

(8) Osborn's Rule

Formulae involving $sinh^2x$ can be obtained from the corresponding trig. formulae by replacing sin^2x with $-sinh^2x$

Also, $sinAsinB$ becomes $-sinhAsinhB$

and tan^2x $(= \frac{sinh^2x}{cosh^2x})$ becomes $-tanh^2x$

If squared terms aren't involved, there is no sign change: $sinx$ becomes $sinhx$.

There is no sign change at all for $coshx$: $cosx$ becomes $coshx$, and cos^2x becomes $cosh^2x$.

Thus, $sinh(A + B) = sinhAcoshB + coshAsinhB$

and $cosh(A + B) = coshAcoshB + sinhAsinhB$

Also $\cosh(2x) = cosh^2x + sinh^2x$

and $cosh^2x - sinh^2x = \cosh(x - x) = 1$

so that $cosh^2x = \frac{1}{2}(\cosh(2x) + 1)$

and $sinh^2x = \frac{1}{2}(\cosh(2x) - 1)$

Proof for $\boldsymbol{cosh(A+B)}$:

As $e^{ix} = cosx + isinx$ & $e^{-ix} = cosx - isinx$,

$cosx = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$ & $sinx = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$

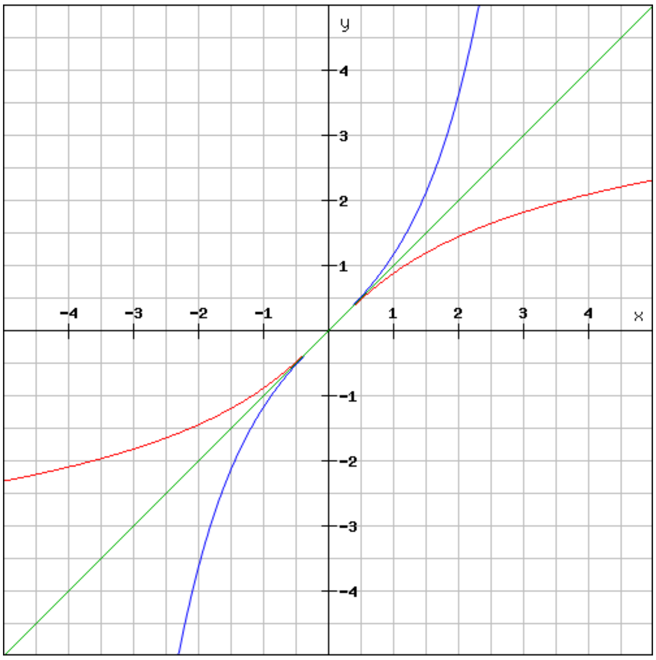
Let $X = ix$ & $Y = iy$

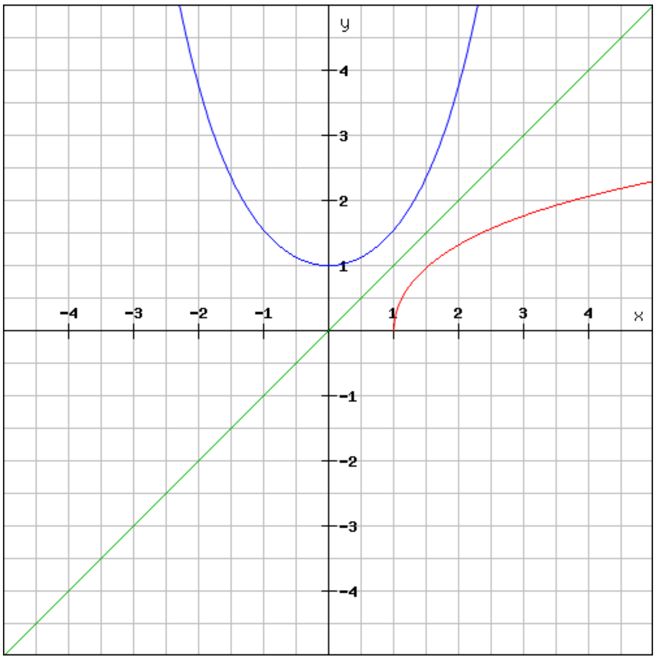
Then $\cosh(X+Y) = \frac{1}{2}\left(e^{i(x+y)} + e^{-i(x+y)}\right) = \cos(x+y)$

$= cosxcosy - sinxsiny = coshXcoshY - \frac{1}{i}sinhX.\frac{1}{i}sinhY$

$= coshXcoshY + sinhXsinhY$

(10) Inverse Functions





$y = arsinhx$ (or $sinh^{-1}x$) $y = arcoshx$ (or $cosh^{-1}x$)

[Note that we write $arsinhx$, and not $arcsinhx$.]

To sketch other inverses ("ar" + ... in all cases):

(i) Either reflect in $y = x$

or reflect in y-axis and rotate by 90° clockwise

(ii) Limit the domain of the original function, if necessary, so that it is 1-1 (and hence the inverse is also 1-1).

(iii) The domain of the inverse will be the range of the original function, and the range of the inverse will be the domain of the original function.

(11) Alternative form of inverse functions

If $y = arsinhx$, then $sinhy = x$

$\Rightarrow \; x = \frac{1}{2}\,(e^y - e^{-y})$

$\Rightarrow 2xe^y = e^{2y} - 1$

$\Rightarrow \; e^{2y} - 2xe^y - 1 = 0$

$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$

$\Rightarrow arsinhx = y = \ln(x + \sqrt{x^2+1})$

(e^y isn't defined for the other root)

Similarly, $arcoshx = \ln(x + \sqrt{x^2-1})$ (see Exercises).

(12) Equations

Possible options for solving equations are:

(i) Convert into an equation (usually quadratic) involving just $sinhx$ or $coshx$

(ii) Use the definition of $sinhx$ or $coshx$ to express the equation in terms of e^x & e^{-x}. A quadratic equation can then often be obtained by multiplying through by e^x.

Example $coshx = 1 \Rightarrow \frac{1}{2}(e^x + e^{-x}) = 1$

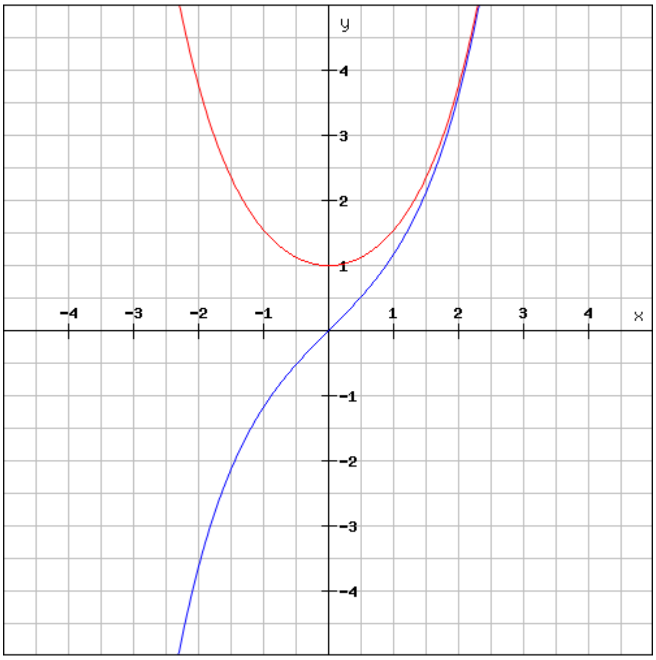
$\Rightarrow e^{2x} + 1 = 2e^x$, which is a quadratic in e^x

(13) Derivatives of $sinhx$ and $coshx$

$\frac{d}{dx}coshx = \frac{d}{dx}\left\{\frac{1}{2}\,(e^x + e^{-x})\right\} = \frac{1}{2}(e^x - e^{-x}) = sinhx$

$$\frac{d}{dx}sinhx = \frac{d}{dx}\left\{\frac{1}{2}\left(e^{x} - e^{-x}\right)\right\} = \frac{1}{2}\left(e^{x} + e^{-x}\right) = coshx$$

We can see from the graphs that the gradient of $coshx$ agrees with the value of $sinhx$, and vice versa.



(14) $\frac{d}{dx}(arsinhx)$

Let $y = arsinhx$, so that $sinhy = x$

Then $\frac{dx}{dy} = coshy$ and $\frac{dy}{dx} = \frac{1}{\sqrt{1+sinh^2y}} = \frac{1}{\sqrt{1+x^2}}$

(Alternatively, $sinhy = x \Rightarrow coshy\frac{dy}{dx} = 1$ etc.)

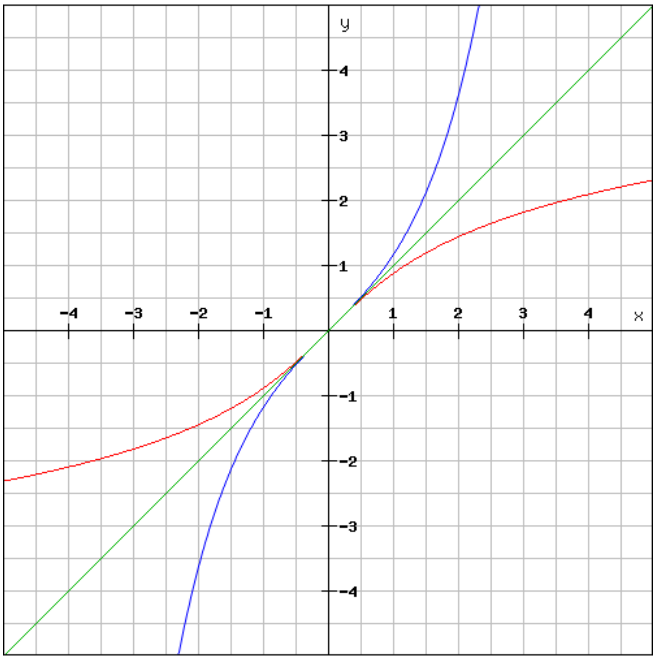
The following features of $\frac{dy}{dx}$, deduced from the formula, can be seen to agree with the graph of $y = arsinhx$ (see red graph below):

(i) always positive

(ii) $\rightarrow 0$ as $x \rightarrow \pm\infty$

(iii) maximum of 1

(at $x = 0$)



[See Exercises for $\frac{d}{dx} arcoshx$]