Hyperbolic Functions – Q9 [Practice/H] (17/6/23)

Show that $arcoshx = \ln\left(x + \sqrt{x^2 - 1}\right)$

Solution

If $y = arcoshx$, then $coshy = x$

$\Rightarrow \quad x = \frac{1}{2}\,(e^y + e^{-y})$

$\Rightarrow 2xe^y = \; e^{2y} + 1$

$\Rightarrow \; e^{2y} - 2xe^y + 1 = 0$

$\Rightarrow e^y = \; \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$

$\Rightarrow y = ln\left(x \pm \sqrt{x^2 - 1}\right)$

However, in order for $y = arcoshx$ to be a function, the negative branch is suppressed (by restricting the domain of $coshx$ to non-negative values). And we can show that $ln\left(x - \sqrt{x^2 - 1}\right) < 0$:

Method 1

Equivalently, we need to show that $x - \sqrt{x^2 - 1} < 1$; or that

$x - 1 < \sqrt{x^2 - 1}$

But $x - 1 = \sqrt{(x - 1)(x - 1)}$ (noting that the range of $coshx$, and hence the domain of $arcoshx$, excludes $x < 1$)

and $\sqrt{(x - 1)(x - 1)} < \sqrt{(x - 1)(x + 1)} = \sqrt{x^2 - 1}$, as required

($y = \sqrt{x}$ is an increasing function,

so $x - 1 < x + 1 \Rightarrow \sqrt{x - 1} < \sqrt{x + 1}$)

[Alternatively, we can argue (slightly informally) that the difference between x^2 and $x^2 - 1$ (ie 1) is contracted by applying the square root function, so that $\sqrt{x^2} - \sqrt{x^2 - 1} < 1$]

Method 2

We expect the unrestricted $y = arcoshx$ to be symmetric about the x-axis (as $y = coshx$ is symmetric about the y-axis). So we could show that $y = ln\left(x \pm \sqrt{x^2 - 1}\right)$ can also be written as

$y = \pm ln\left(x + \sqrt{x^2 - 1}\right)$, and then reject the negative branch as before.

So we want to show that $ln\left(x - \sqrt{x^2 - 1}\right) = -ln\left(x + \sqrt{x^2 - 1}\right)$:

RHS $= ln\left(\frac{1}{x+\sqrt{x^2-1}}\right) = ln\left(\frac{x-\sqrt{x^2-1}}{x^2-(x^2-1)}\right) = ln\left(x - \sqrt{x^2 - 1}\right)$, as required