Hyperbolic Functions – Q10 [Practice/M] (17/6/23)

Derive an expression for $arsinh(a)$ in the form lnb

Solution

Let $x = arsinh(a)$, so that $sinhx = a$

and $\frac{1}{2}(e^x - e^{-x}) = a$

Then $\frac{1}{2}(e^{2x} - 1) = ae^x$

and $e^{2x} - 2ae^x - 1 = 0$,

so that $e^x = \frac{2a \pm \sqrt{4a^2+4}}{2} = a + \sqrt{a^2+1}$ (rejecting the negative root)

Thus $arsinh(a) = \ln\left(a + \sqrt{a^2+1}\right)$

(noting that $a + \sqrt{a^2+1} > 0$)

[Note that $arsinh\left(\frac{x}{a}\right) = \ln\left(\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 + 1}\right)$

$= \ln\left(\frac{x+\sqrt{x^2+a^2}}{a}\right) = \ln\left(x + \sqrt{x^2+a^2}\right) - lna$

In the formulae booklets, $\int \frac{1}{\sqrt{a^2+x^2}} dx$ is often given as

"$arsinh\left(\frac{x}{a}\right)$ or $\ln\left(x + \sqrt{x^2+a^2}\right)$" but, as we've just seen, these two expressions differ by a constant.]