Hyperbolas - Exercises (Solutions) (4 pages; 18/8/19)

(1) Show that the equation of the tangent to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(acosht, bsinht)$ is

$yasinht = xbcosht - ab$

Solution

Using the parametric equations $x = acosht$ & $y = bsinht$,

$\frac{dx}{dt} = asinht$ & $\frac{dy}{dt} = bcosht$,

so that $\frac{dy}{dx} = \frac{bcosht}{asinht}$

and the equation of the tangent at $(acosht, bsinht)$ is

$\frac{y-bsinht}{x-acosht} = \frac{bcosht}{asinht}$

and hence $yasinht - absinh^2t = xbcosht - abcosh^2t$,

so that $yasinht = xbcosht - ab$

(2) Given that the tangent in (C)(1) meets the asymptotes of the hyperbola at the points P & Q, show that the mid-point of P & Q is $(acosht, bsinht)$.

Solution

The asymptotes of the hyperbola are $y = \pm\frac{b}{a}x$

From (1), the tangent to the hyperbola at $(acosht, bsinht)$ meets the asymptote $y = \frac{b}{a}x$ at P (say), where $bxsinht = xbcosht - ab$ and the asymptote $y = -\frac{b}{a}x$ at Q where

$-bxsinht = xbcosht - ab$

so that P is the point $(\frac{a}{cosht-sinht}, \frac{b}{cosht-sinht})$

and Q is the point $(\frac{a}{cosht+sinht}, \frac{-b}{cosht+sinht})$

The mid-point of P & Q is therefore

$(\frac{1}{2}\left[\frac{a}{cosht-sinht}+\frac{a}{cosht+sinht}\right], \frac{1}{2}\left[\frac{b}{cosht-sinht}+\frac{-b}{cosht+sinht}\right])$

$=\left(\frac{acosht}{cosh^2t-sinh^2t}, \frac{bsinht}{cosh^2t-sinh^2t}\right)=(acosht, bsinht)$, as required.

(3) In the case where $b=a$, find the area of the triangle OPQ (where P & Q are as in (C)(2), and O is the Origin).

Solution

The two asymptotes are at right angles to each other, so that the required area, $A=\frac{1}{2}OP.OQ$

Then $4A^2=\left(\left(\frac{a}{cosht-sinht}\right)^{\mathbf{2}}+\left(\frac{a}{cosht-sinht}\right)^{\mathbf{2}}\right)$

$\times\left(\left(\frac{a}{cosht+sinht}\right)^{\mathbf{2}}+\left(\frac{-a}{cosht+sinht}\right)^{\mathbf{2}}\right)$

$=\left(\frac{2a^2}{(cosht-sinht)^2}\right)\left(\frac{2a^2}{(cosht+sinht)^2}\right)$

$=\frac{4a^4}{(cosh^2t-sinh^2t)^2}=4a^4$

and therefore $A=a^2$

(4) The chord PQ, where P and Q are points on the rectangular hyperbola $xy=c^2$, has gradient 1. Show that the locus of the point of intersection of the tangents from P and Q is the line

$y=-x$. [Edx FP3 textbook, Ex. 2G, Q9]

Solution

Let P & Q be the points $\left(ct_1, \frac{c}{t_1}\right)$ & $\left(ct_2, \frac{c}{t_2}\right)$, respectively.

As the gradient of PQ is 1, $\frac{\frac{c}{t_2}-\frac{c}{t_1}}{ct_2-ct_1} = 1$, so that

$\frac{1}{t_2} - \frac{1}{t_1} = t_2 - t_1$

$\Rightarrow \frac{t_1-t_2}{t_1 t_2} = t_2 - t_1$

$\Rightarrow t_1 t_2 = -1$, as $t_1 \neq t_2$ (P & Q being distinct points)

The equation of the tangent from P is:

$\frac{y-\frac{c}{t_1}}{x-ct_1} = \frac{dy/dt}{dx/dt}|t = t_1$, where $x = ct$ & $y = \frac{c}{t}$

so that $dy/dt = -\frac{c}{t^2}$ & $dx/dt = c$

and the equation of the tangent from P is

$\frac{y-\frac{c}{t_1}}{x-ct_1} = \frac{\left(-\frac{c}{t_1{}^2}\right)}{c} \Rightarrow t_1{}^2 y - t_1 c = -(x - ct_1)$

$\Rightarrow t_1{}^2 y = -x + 2ct_1 \quad (1)$

Similarly, the equation of the tangent from Q is $t_2{}^2 y = -x + 2ct_2$

and these lines intersect where

$t_1{}^2 y - 2ct_1 = t_2{}^2 y - 2ct_2$,

so that $y(t_1{}^2 - t_2{}^2) = 2c(t_1 - t_2)$

and $y = \frac{2c}{t_1+t_2}$ (as $t_1 \neq t_2$)

Then, from (1), $x = 2ct_1 - \frac{2ct_1{}^2}{t_1+t_2}$

$= \frac{2ct_1{}^2+2ct_1t_2-2ct_1{}^2}{t_1+t_2}$

$= \frac{2ct_1t_2}{t_1+t_2}$

and so $\frac{y}{x} = \frac{1}{t_1t_2} = -1$ (found earlier),

and thus the points of intersection satisfy $y = -x$, as required.