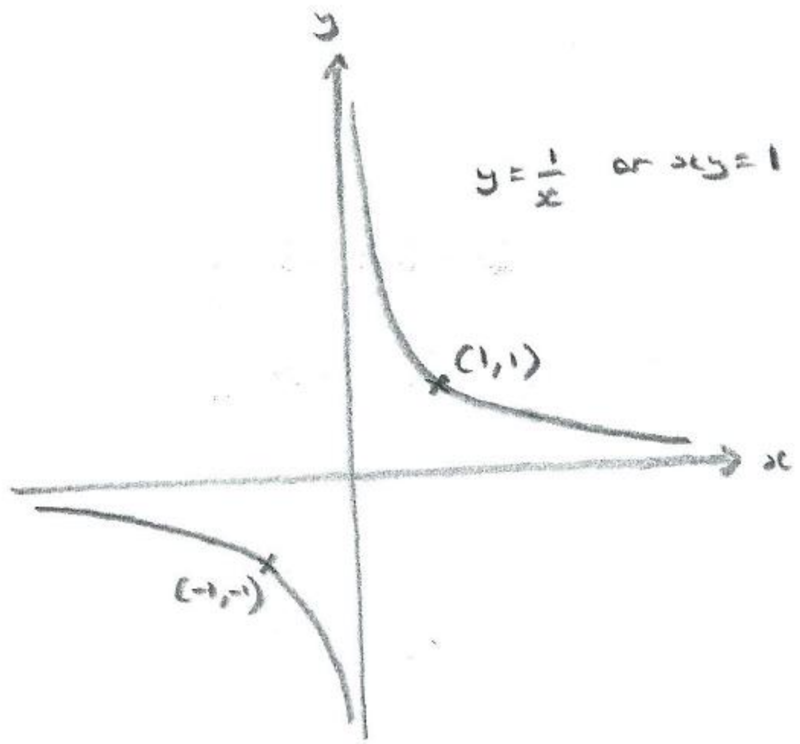
Hyperbolas (13 pages; 12/8/19)

See "Conics" for features that are common to parabolas, ellipses and hyperbolas (as well as circles).

(1) The simplest type of hyperbola is the rectangular hyperbola $y = \frac{1}{x}$ or $xy = 1$



A rectangular hyperbola is defined as having perpendicular asymptotes.

Other hyperbolas can be obtained by stretching , rotating, reflecting and translating $xy = 1$, as will be seen.

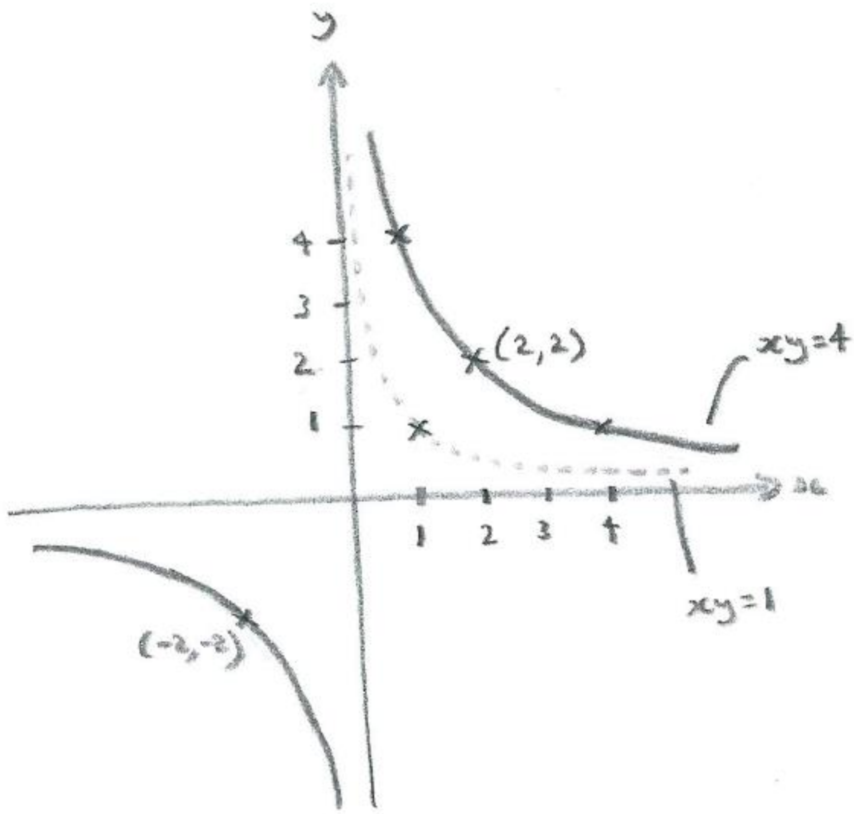
(2) Consider the effect of the following transformations:

(i) stretch of scale factor 4 in the x direction

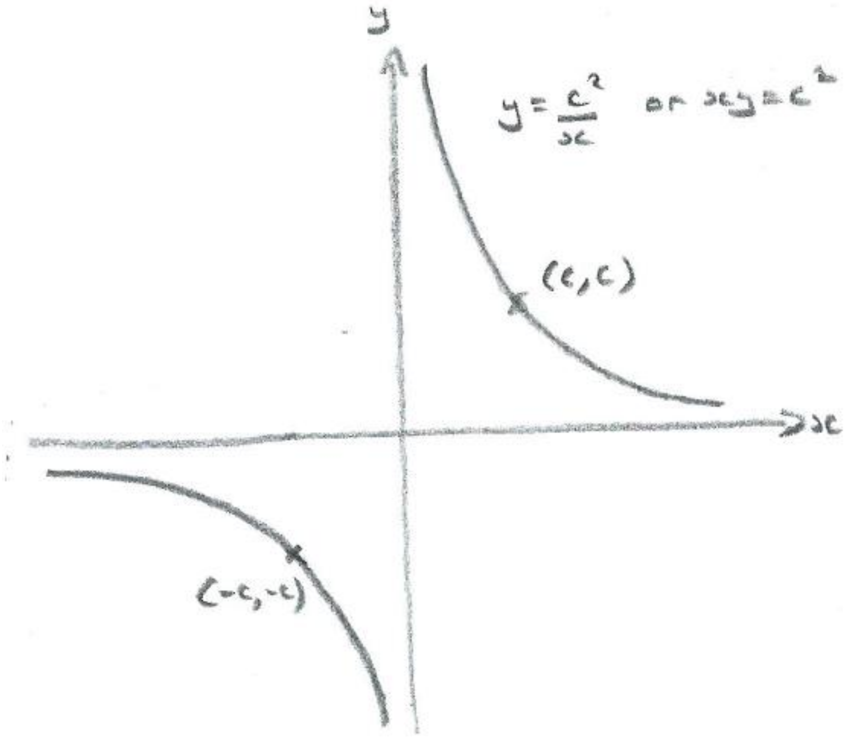
x is replaced by $\frac{x}{4}$, so that $xy = 1 \rightarrow \left(\frac{x}{4}\right)y = 1$; ie $xy = 4$

(ii) stretch of scale factor 4 in the y direction

similarly, $xy = 1 \rightarrow x\left(\frac{y}{4}\right) = 1$; ie $xy = 4$



So the hyperbola $xy = c^2$ can be obtained from $xy = 1$ by a stretch in either the x direction or y direction (or a combination of the two). [The use of the square rules out a negative stretch; ie a reflection in the x or y axis.]



(3) Rectangular hyperbola with asymptotes of $y = \pm x$

Suppose that the point (x, y) on the hyperbola $xy = c^2$ is rotated by $\frac{\pi}{4}$ radians clockwise, to give the point (u, v).

Then $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} cos\left(-\frac{\pi}{4}\right) & -sin\left(-\frac{\pi}{4}\right) \\ sin\left(-\frac{\pi}{4}\right) & cos\left(-\frac{\pi}{4}\right) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

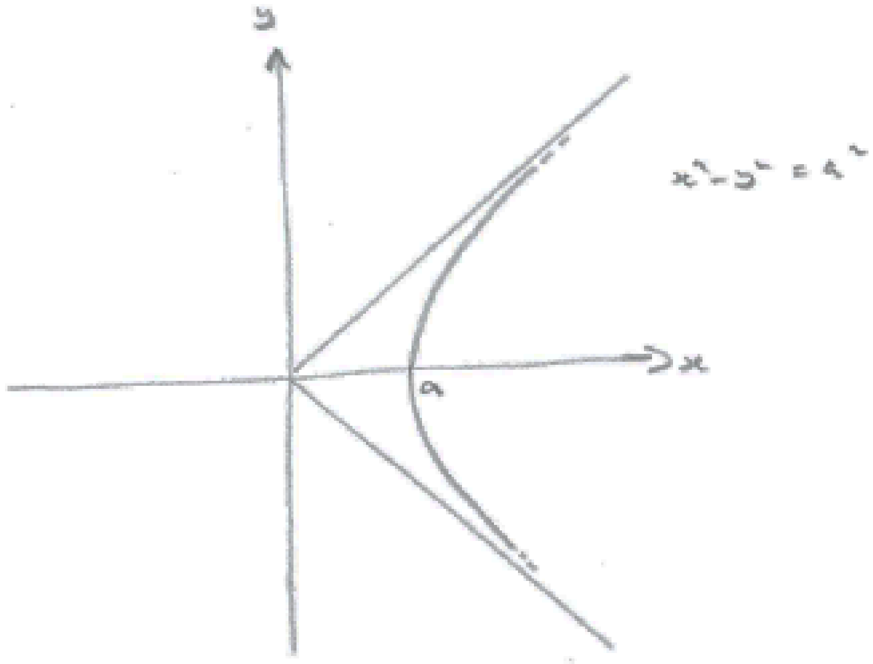
and $\begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} u \\ v \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

so that $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} u - v \\ u + v \end{pmatrix}$

Hence $xy = c^2 \Rightarrow \frac{1}{\sqrt{2}}(u - v)\frac{1}{\sqrt{2}}(u + v) = c^2$,

so that $u^2 - v^2 = 2c^2$

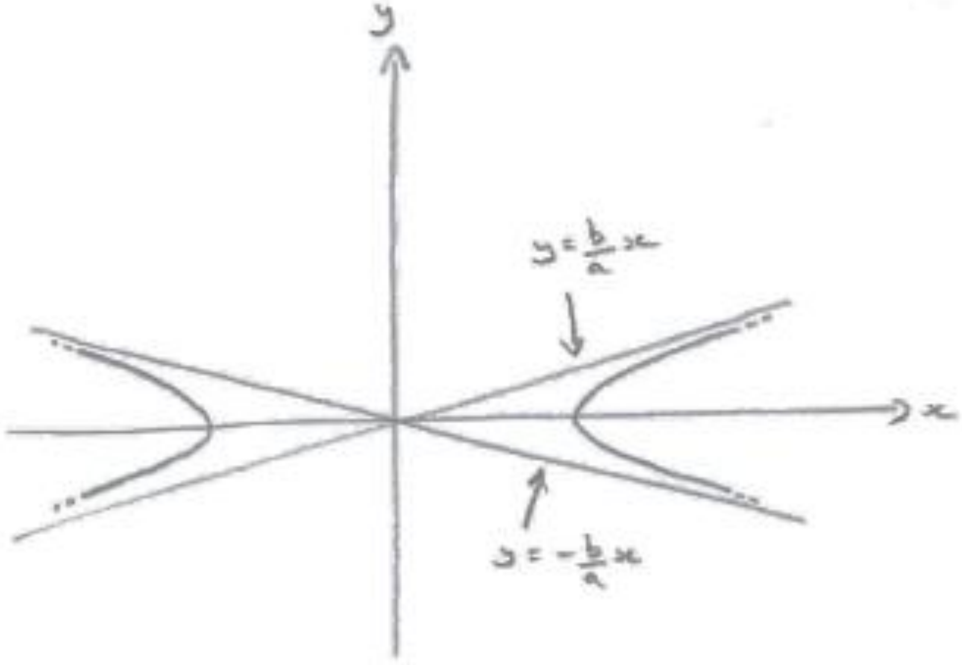
On relabelling this gives the rectangular hyperbola of the form $x^2 - y^2 = a^2$ or $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$, the asymptotes of which are $y = \pm x$



(4) Hyperbola with asymptotes of $y = \pm\frac{b}{a}x$

If a stretch of scale factor $\frac{b}{a}$ is applied in the y direction, then the curve $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$ is transformed to $\frac{x^2}{a^2} - \frac{\left(\frac{ay}{b}\right)^2}{a^2} = 1$,

or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, and this is a hyperbola with asymptotes of $y = \pm\frac{b}{a}x$ (it is no longer rectangular).



Apart from any question of translation, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the most general type of hyperbola.

(5) The asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be established directly as follows.

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 1$ (A)

As x & $y \to \infty$, $\left(\frac{x}{a} + \frac{y}{b}\right) \to \infty$, so that $\left(\frac{x}{a} - \frac{y}{b}\right) \to 0$, in order for (A) to hold.

So one asymptote is $y = \frac{b}{a}x$

As $x \to \infty$ & $y \to -\infty$, $\left(\frac{x}{a} - \frac{y}{b}\right) \to \infty$, so that

$\left(\frac{x}{a} + \frac{y}{b}\right) \to 0$; again, in order for (A) to hold.

So the other asymptote is $y = -\frac{b}{a}x$

(6) Other variations

The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be described as being centred on the Origin, with vertical directrices (at $x = \pm\frac{a}{e}$ [see below]). A reflection in $y = x$ produces the form with horizontal directrices: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

In addition, there is always the option of translating any curve by $\binom{x_0}{y_0}$, by replacing x with $x - x_0$, and y with $y - y_0$.

(7) Parametric equations:

General hyperbola

Cartesian equation: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Parametric equations:

Let $x = asect$, $y = btant$ $(a, b > 0; 0 \leq t < 2\pi)$,

as $sec^2t - tan^2t = 1$

The equations

$x = acosht$, $y = bsinht$ (for $x > 0$)

$x = -acosht$, $y = bsinht$ (for $x < 0$) , with $t \in \mathbb{R}$ in both cases,

can also be used if the two branches are to be treated separately (eg motion of a comet).

Rectangular hyperbola, with horizontal & vertical asymptotes

Cartesian equation: $xy = c^2$

Parametric equations:

Let $x = ct$, $y = \frac{c}{t}$, where $c > 0$ and $t \in \mathbb{R}$ $(t \neq 0)$.

Then $xy = ct\left(\frac{c}{t}\right) = c^2$

(8) For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, suppose that e is defined by $b^2 = a^2(e^2 - 1)$, and that one focus is taken to be the point $(0, ae)$, whilst one directrix is taken to be the line $x = \frac{a}{e}$

Exercise: Show that $e = \frac{PS}{PM}$, where P is the general point (x, y), with S and M being defined as in the previous "Conics", section (2).

Solution

$$\left(\frac{PS}{PM}\right)^2 = \frac{(x-ae)^2+y^2}{\left(x-\frac{a}{e}\right)^2}$$

Using the parametric form $x = asect$, $y = btant$,

$$\frac{(x-ae)^2+y^2}{\left(x-\frac{a}{e}\right)^2} = \frac{a^2(sect-e)^2+a^2(e^2-1)tan^2t}{a^2\left(sect-\frac{1}{e}\right)^2}$$

$$= \frac{e^2[(sect-e)^2+\left(e^2-1\right)tan^2t]}{(esect-1)^2} = \frac{e^2[sec^2t-2esect+e^2+e^2tan^2t-tan^2t]}{(esect-1)^2}$$

$$= \frac{e^2[1-2esect+e^2sec^2t]}{(esect-1)^2} = e^2$$

Also, from $b^2 = a^2(e^2 - 1)$, we can see that $e > 1$.

It can be shown similarly that the focus could also be taken to be the point $(0, -ae)$, with the directrix being the line $x = -\frac{a}{e}$

(9) Eccentricity of Rectangular Hyperbola

In this case $b = a$ (so that, with vertical directrices, $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$), and since $b^2 = a^2(e^2 - 1)$, $1 = e^2 - 1$, and hence $e = \sqrt{2}$

(10) Tangents and Normals: Rectangular hyperbola, with horizontal & vertical asymptotes

Exercise Find the equations of the tangents and normals to the rectangular hyperbola $xy = 4$ at the point (4,1).

Solution

$y = \frac{4}{x} = 4x^{-1} \Rightarrow \frac{dy}{dx} = -4x^{-2}$

When $x = 4$, $\frac{dy}{dx} = -\frac{1}{4}$

Equation of tangent:

$\frac{y-1}{x-4} = -\frac{1}{4}$

$\Rightarrow y - 1 = -\frac{x}{4} + 1$

$\Rightarrow y = 2 - \frac{x}{4}$

The equation of the normal is $\frac{y-1}{x-4} = 4$ etc

Using the parametric form for the more general $xy = c$:

$x = ct, \; y = \frac{c}{t}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-ct^{-2}}{c} = -t^{-2}$

Equation of tangent: $\frac{y-\frac{c}{t}}{x-ct} = -t^{-2}$

$\Rightarrow y - \frac{c}{t} = -\frac{x}{t^2} + \frac{c}{t}$

$\Rightarrow y = \frac{2c}{t} - \frac{x}{t^2}$

or $yt^2 = 2ct - x$

$\Rightarrow x + t^2 y = 2ct$

(11) Tangents and Normals (general hyperbola)

For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we can obtain the equation of the tangent via either the Cartesian or the parametric forms:

Cartesian

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} - \frac{2y}{b^2} . \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{x/a^2}{y/b^2} = \frac{b^2 x}{a^2 y}$

Then the tangent at the point (x_0, y_0) is $\frac{y-y_0}{x-x_0} = \frac{b^2 x_0}{a^2 y_0}$

[Note that (x, y) now relates to a general point on the tangent, rather than the hyperbola.]

This can also be rewritten as $yy_0 a^2 - {y_0}^2 a^2 = xx_0 b^2 - {x_0}^2 b^2$

and hence $\frac{yy_0}{b^2} - \frac{{y_0}^2}{b^2} = \frac{xx_0}{a^2} - \frac{{x_0}^2}{a^2}$

or $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = \frac{{x_0}^2}{a^2} - \frac{{y_0}^2}{b^2}$

As (x_0, y_0) lies on the hyperbola, $\frac{{x_0}^2}{a^2} - \frac{{y_0}^2}{b^2} = 1$, so we have $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$ as the equation of the tangent.

Parametric

$x = asect\ , y = btant,$

so that $\frac{dx}{dt} = a\frac{d}{dt}(cost)^{-1} = -a(cost)^{-2}(-sint) = asec^2t.sint$

and $\frac{dy}{dt} = bsec^2t$

Then $\frac{dy}{dx} = \frac{dy/_{dt}}{dx/_{dt}} = \frac{bsec^2t}{asec^2t.sint} = \frac{bcosect}{a}$, and the equation of the tangent is

$\frac{y-btant}{x-asect} = \frac{bcosect}{a}$

$\Rightarrow ay - abtant = bcosect.x - ab.cosect.sect$

$\Rightarrow aytant - abtan^2t = bsect.x - absec^2t$

[multiplying both sides by $tant$; not an obvious step, admittedly]

$\Rightarrow aytant + ab = bsect.x$

$\Rightarrow \frac{xsect}{a} - \frac{ytant}{b} = 1$

[compare with the equivalent form for the ellipse:

$\frac{xcost}{a} + \frac{ysint}{b} = 1$; see "Ellipses"]

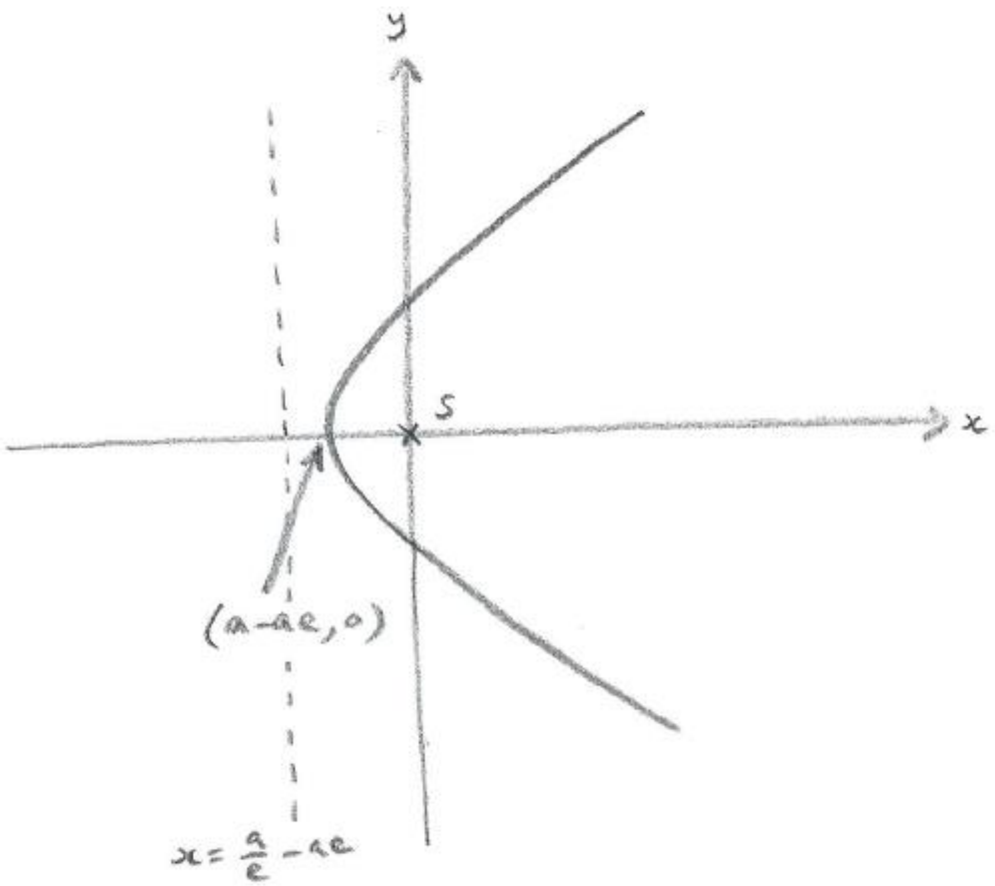
[As a check: $\frac{b^2x_0}{a^2y_0} = \frac{b^2asect}{a^2btant} = \frac{bsectcost}{asint} = \frac{bcosect}{a}$]

[If the tangent is to have a specified gradient (m), then we can find the (in general, two) possible points on the hyperbola by proceeding as if to find the intersection of the straight line,

$y = mx + c$, with the hyperbola, and choosing c such that the discriminant of the resulting quadratic is zero (so that the line meets the hyperbola as a tangent).]

(12) Polar form of hyperbola

See "Conics" for the derivation of the polar form of a general conic: $r = \frac{ep}{1-ecos\theta}$, where p is the (positive) distance between the focus and the directrix, for the case where the directrix is vertical and lies to the left of the pole.



In order for the hyperbola to have the same location and orientation as the general conic used to derive the above formula, we make a translation of ae to the left (so that the right-hand focus is moved to the Origin).

For this hyperbola, $p = ae - \frac{a}{e}$, so that

$r = \frac{ep}{1-ecos\theta}$ becomes $r = \frac{a(e^2-1)}{1-ecos\theta}$

It can be seen (by trying out specific values of θ) that this equation only represents the right-hand branch of the hyperbola.

To obtain the left-hand branch, we need to position the left-hand focus at the origin, and this gives the equation

$r = \frac{a(e^2-1)}{1+ecos\theta}$

Exercise

(1) Use matrices to show that the rectangular hyperbola $x^2 - y^2 = a^2$ can be obtained by rotating the rectangular hyperbola $xy = c^2$, expressing a^2 in terms of c.

Solution

The asymptotes of $x^2 - y^2 = a^2$ are $y = \pm x$, whilst the asymptotes of

$xy = c^2$ are the x and y axes.

So consider a rotation of 45° clockwise.

Then the point (x, y) on the hyperbola $xy = c^2$ is transformed to the point (u, v), where

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} cos(-45°) & -sin(-45°) \\ sin(-45°) & cos(-45°) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$= \frac{1}{\sqrt{2}}\begin{pmatrix} x+y \\ -x+y \end{pmatrix}$

Then $u^2 - v^2 = (u-v)(u+v)$

$= \frac{1}{\sqrt{2}}(2x).\frac{1}{\sqrt{2}}(2y) = 2xy = 2c^2$

Relabelling gives $x^2 - y^2 = 2c^2$ (and $a^2 = 2c^2$).