Hyperbolas Q6 [9 marks](2/7/21)

Exam Boards

OCR : -

MEI: -

AQA: -

Edx: Further Pure 1 (Year 2)

Suppose that P is a general point on a rectangular hyperbola and that the tangent at P crosses the x and y axes at A and B respectively. Show that:

(i) $AP = BP$ [7 marks]

(ii) the triangle OAB has a constant area, as P varies [2 marks]

Solution

(i) Suppose that the equation of the rectangular hyperbola is

$xy = c^2$ and that P is the point $\left(a, \frac{c^2}{a}\right)$. [1 mark]

Then $\frac{dy}{dx} = -c^2x^{-2}$, [1 mark]

and the equation of the tangent at P is

$y - \frac{c^2}{a} = -\frac{c^2}{a^2}(x - a)$ [1 mark]

At A, $0 - \frac{c^2}{a} = -\frac{c^2}{a^2}(x - a)$, so that $a = x - a$ and $x = 2a$ [1 mark]

At B, $y - \frac{c^2}{a} = -\frac{c^2}{a^2}(0 - a)$, so that $y = \frac{c^2}{a} + \frac{c^2}{a} = \frac{2c^2}{a}$ [1 mark]

Then $AP^2 = (2a - a)^2 + (\frac{c^2}{a} - 0)^2 = a^2 + \frac{c^4}{a^2}$ [1 mark]

and $BP^2 = (a - 0)^2 + (\frac{2c^2}{a} - \frac{c^2}{a})^2 = a^2 + \frac{c^4}{a^2}$

Thus $AP = BP$, as required. [1 mark]

(ii) Area of OAB $= \frac{1}{2}(2a)\left(\frac{2c^2}{a}\right) = 2c^2$; ie a constant [2 marks]