

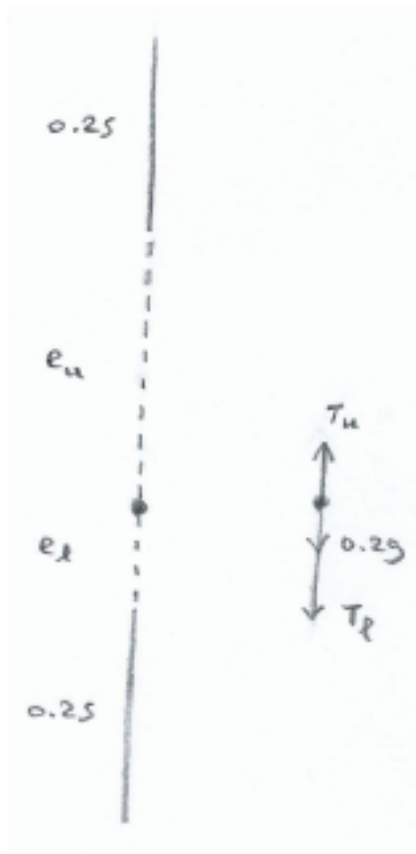
Hooke's Law - Exercises (Solutions)

(7 pages; 30/1/19)

(1) A particle of mass 200g is attached at the mid-point of an elastic string of natural length 0.5m and modulus of elasticity λ , which hangs vertically between two points, 1m apart.

- (i) How far will the particle be below the top point if $\lambda = 1$?
- (ii) Determine the minimum value of λ such that there is no slack in the string.

Solution



(i) Let the extensions of the upper and lower parts of the string be e_u and e_l , respectively, and the tensions in the two parts T_u and T_l .

Then, referring to the diagram,

$$T_u = \frac{\lambda e_u}{0.25} \quad , \quad T_l = \frac{\lambda e_l}{0.25} \quad (\text{assuming the string is not slack})$$

$$\text{Equilibrium} \Rightarrow T_u = T_l + 0.2g \quad ; \text{ also } e_u + e_l = 0.5 \quad (1)$$

$$\text{Hence } \lambda e_u = \lambda e_l + 0.05g$$

$$\text{and so } \lambda e_u = \lambda(0.5 - e_u) + 0.05g,$$

$$\text{giving } 2\lambda e_u = 0.5\lambda + 0.05g$$

$$\text{and hence } e_u = \frac{0.5\lambda + 0.05g}{2\lambda} \quad (2)$$

$$\text{Thus when } \lambda = 1, e_u = 0.495$$

$$\text{and the distance below the top point is } 0.25 + 0.495 = 0.745m$$

(ii) The string is slack if $e_l < 0$

$$\text{From (1) \& (2), } e_l = 0.5 - 0.25 - \frac{0.025g}{\lambda} = 0.25 - \frac{0.025g}{\lambda}$$

$$\text{Thus we require } 0.25 - \frac{0.025g}{\lambda} \geq 0,$$

$$\text{so that } 0.25 \geq \frac{0.025g}{\lambda} \quad \text{and } \lambda \geq 0.1g = 0.98$$

(2) A particle of mass 200g hangs at a point Q, suspended from a fixed point P, by means of a spring of original length 20cm and modulus of elasticity 5N. It is pulled down to a point R, which is 35cm below P. The particle is then released.

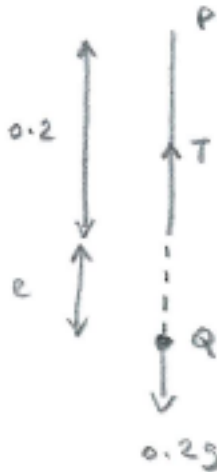
Ignoring any resistances to motion, find:

(i) the work done in pulling the particle down to R

(ii) the maximum speed of the particle after it is released, and the point at which this occurs

(iii) the distance of the particle below P when it reaches its maximum height, at position S, and show that the distance QS equals the distance QR

Solution



[Note: The g in the diagram (gravity) is not to be confused with g for grams.]

(i) If e is the extension of the spring at Q (in metres),

$$\text{Hooke's Law} \Rightarrow \frac{\lambda e}{l} = T = mg \Rightarrow \frac{5e}{0.2} = (0.2)(9.8) \Rightarrow e = 0.0784$$

Taking the zero of gravitational potential energy (GPE) to be R, the total energy of the particle at Q is:

GPE + EPE + KE (where EPE is elastic potential energy & KE is kinetic energy)

$$= (0.2)(9.8)(0.35 - 0.2 - 0.0784) + \frac{1}{2} \left(\frac{5}{0.2} \right) (0.0784)^2 + 0$$

$$= 0.140336 + 0.076832 + 0 = 0.217168$$

The total energy of the particle at R is:

$$0 + \frac{1}{2} \left(\frac{5}{0.2} \right) (0.15)^2 + 0 = 0.28125$$

Thus the work done = $0.28125 - 0.217168 = 0.064082 = 0.0641 \text{ J}$ (3sf)

(ii) The maximum speed will occur when the particle is not accelerating; ie at Q, where the net force on the particle is zero [as $T = mg$ at the equilibrium position].

The KE of the particle at Q will equal the work done to pull it down to R, as this is the energy gained by the particle since it was last at Q.

Hence $\frac{1}{2} (0.2)v^2 = 0.064082$ (where v is the maximum speed)

and $v = 0.80051 = 0.801 \text{ ms}^{-1}$ (3sf)

(iii) Let d be the distance below P when the particle is at S.

The total energy of the particle at S is:

$$(0.2)(9.8)(0.35 - d) + \frac{1}{2} \left(\frac{5}{0.2} \right) (d - 0.2)^2 + 0$$

and this equals the energy at R of 0.28125, so that

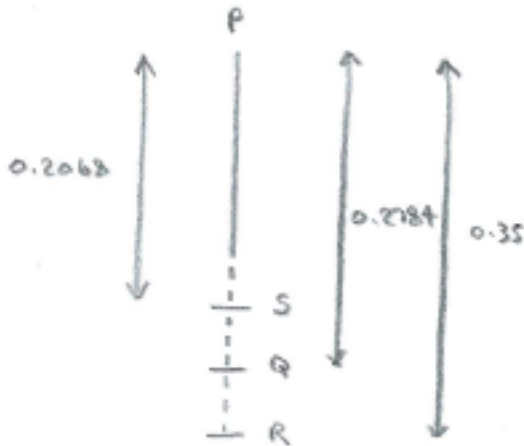
$$12.5d^2 - 6.96d + 0.90475 = 0$$

$$\text{and } d = \frac{6.96 \pm \sqrt{3.2041}}{25} = 0.35 \text{ or } 0.2068$$

Thus 0.35 corresponds to R and S is the point 20.68cm below P.

This is $20 + 7.84 - 20.68 = 7.16$ cm above the equilibrium position Q, whilst R is $35 - (20 + 7.84) = 7.16$ cm below Q.

[The particle oscillates between R and S.]



(3) A bungee jumper of mass 80kg is attached to a rope of original length 10m and modulus of elasticity 1600N. How far will he or she fall? (Take $g=10$)

Solution

Let e be the extension of the rope.

Gain in elastic PE = loss of gravitational PE, so that

$$\frac{1}{2} \left(\frac{1600}{10} \right) e^2 = 80(10)(10 + e)$$

$$\Rightarrow e^2 = 100 + 10e$$

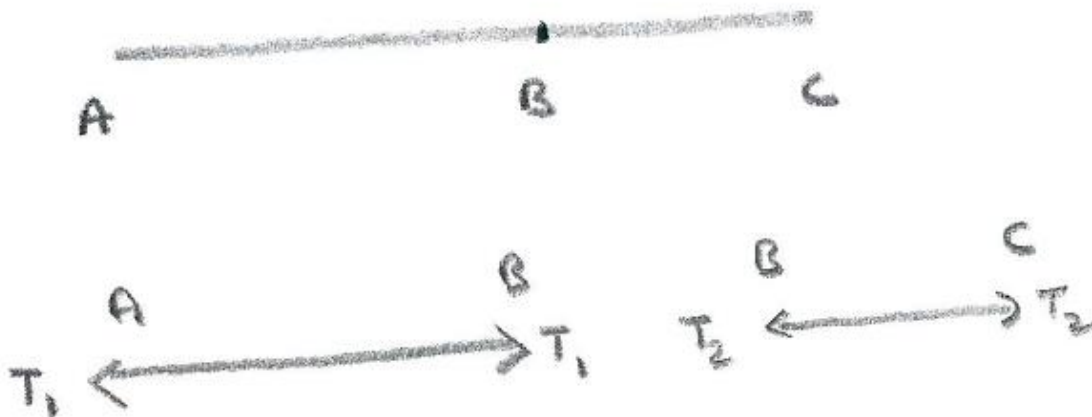
$$\Rightarrow e^2 - 10e - 100 = 0$$

$$\Rightarrow e = \frac{10 \pm \sqrt{100 + 400}}{2} = 16.18\text{m (ignoring -ve value)}$$

So bungee jumper falls by $10 + 16.18 = 26.18\text{m}$

(4) Two elastic strings AB and BC are joined together at B, to form one long string. String AB has natural length $4m$ and modulus of elasticity $20N$; string BC has natural length $2m$ and modulus of elasticity $30N$. The ends A and C of the long string are attached to two fixed points which are $10m$ apart. Find the tension in the combined string.

Solution



Considering the force diagram for AB: by N2L, the reaction at A will equal the force applied by string BC at B. Call this T_1 - this is the tension that AB is under. Similarly, string BC will be under tension T_2 . By N3L, the forces that the two strings apply to each other will be equal and opposite, so that $T_1 = T_2 = T$, say. The tension in the combined string (determined by the reactions at A and C) will therefore be T also.

For string AB, Hooke's law $\Rightarrow T_1 = \frac{20e_1}{4}$, where e_1 is the extension of string AB.

Similarly, for string BC, $T_2 = \frac{30e_2}{2}$

Also $(4 + e_1) + (2 + e_2) = 10$, so that $e_1 + e_2 = 4$

Then $T_1 = T_2 \Rightarrow \frac{20e_1}{4} = \frac{30e_2}{2}$, so that $5e_1 = 15(4 - e_1)$,

and $e_1 = 3(4 - e_1)$, so that $4e_1 = 12$ and hence $e_1 = 3$, and $e_2 = 1$.

Therefore $T = 5e_1 = 15N$

Note: A result [which would probably have to be proved in an exam, though possibly not for STEP] that can be applied here is that, with strings of stiffness k_1 and k_2 in series, the combined string has stiffness k given by $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$

In this case, $\frac{1}{k} = \frac{4}{20} + \frac{2}{30} = \frac{16}{60}$, so that $k = \frac{15}{4}$,

and $T = k(e_1 + e_2) = \frac{15}{4}(4) = 15N$