# Hooke's Law (9 pages; 26/1/19)

(1) Hooke's law can be expressed in 3 ways:

(i) $T = \frac{EAx}{l}$ , where $E$ is Young's modulus and $A$ is the cross-sectional area of a string (or wire/spring)

$E$ depends only on the nature of the material, and has S.I. units of $Nm^{-2}$ or Pascals (Pa).

This version is used in Physics, but not in A Level Mechanics.

(ii) $T = \frac{\lambda x}{l}$ , where $\lambda$ is the modulus of elasticity (of the string)

As well as the nature of the material, $\lambda$ depends on the cross-sectional area of the string. Its S.I. units are $N$.

This is the version most commonly used in A Level Mechanics.

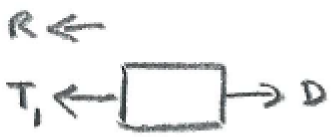
(iii) $T = kx$ , where $k$ is the stiffness (of the string)

As well as the nature of the material and the cross-sectional area of the string, $k$ depends on the original length of the string. In other words, $k$ is specific to each piece of string. Its S.I. units are $Nm^{-1}$.

## (2) Tension in strings (etc)

(2.1) Consider a car pulling a caravan, by means of a towbar.

Separate force diagrams can be drawn for the car and the towbar.

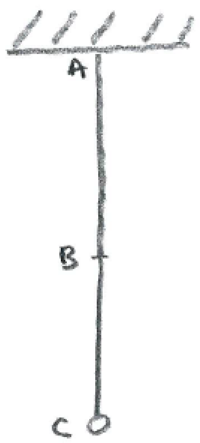



For the towbar, the fact that it is under tension means that the external forces at its ends are pulling on the towbar (if the towbar is under compression, then the external forces are pressing on the ends). Initially these forces are taken to be $T_1$ and $T_2$. By Newton's 2nd law, $T_1 - T_2 = ma$, where $m$ is the mass of the towbar and $a$ its acceleration.

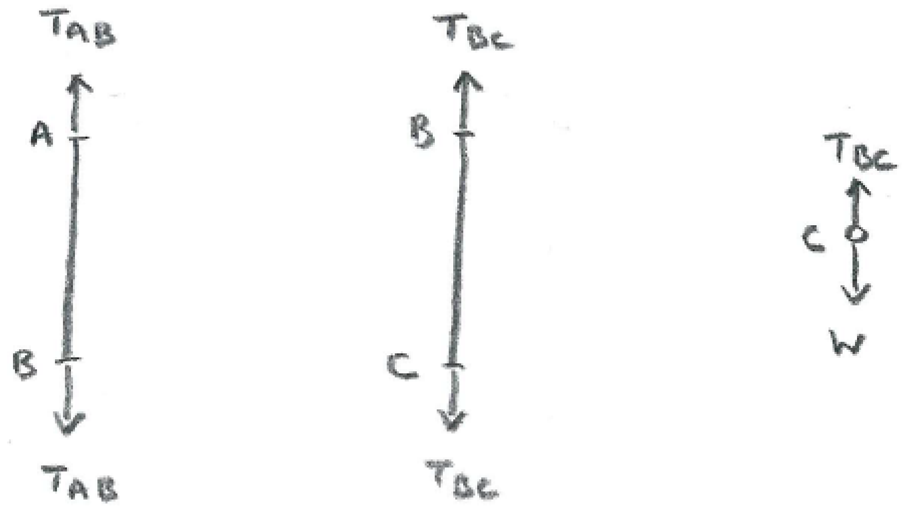
Then, if either $m$ is deemed to be negligible, or if $a = 0$, then approximately: $T_1 = T_2 = T$, say.

By Newton's 3rd law, as the car is pulling on the towbar with force $T$, it follows that the towbar is also pulling on the car with a force $T$.

(2.2) Consider the situation shown below, where the string AB (with original length $l_1$, modulus of elasticity $\lambda_1$ and extension $x_1$) is hanging vertically from a fixed point A, and is connected to the string BC (with original length $l_2$, modulus of elasticity $\lambda_2$ and extension $x_2$), which also hangs vertically and has a load of weight W at C.



Force diagrams can be drawn for the string AB, the string BC, and the load at C.




The external forces at the ends of AB and BC follow from the definition of tension. By Newton's 3rd law, the force on AB at B (ie $T_{AB}$) will be equal (but with opposite direction) to the force on BC at B (ie $T_{BC}$), so that $T_{AB} = T_{BC}$. This is a standard result that applies when strings etc with different $\lambda s$ are connected together vertically or horizontally. However, it won't apply in more complicated situations, such as that in (2.4).

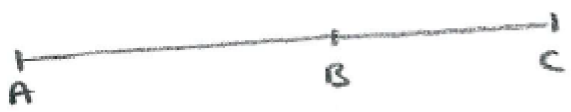
In the force diagram for C, the upward force is $T_{BC}$, by Newton's 3rd law. As the system is at rest, $W - T_{BC} = 0$, by Newton's 2nd law. Thus $T_{AB} = T_{BC} = W$, and so $\frac{\lambda_1 x_1}{l_1} = \frac{\lambda_2 x_2}{l_2} = W$

Note that we cannot bypass the load at C by saying that the downward force on BC is W, because W is the gravitational force on the load C; not on the string. Instead $T_{BC}$ has to be introduced as the reaction force between the string and the load.

When the system is at rest (or in equilbrium; ie when the acceleration is zero), the effect is the same as if a force of W were applied directly to the string. But if the system has a non-zero acceleration (so that $W \neq T_{BC}$), then this would not be the case.

The string could of course be pulled on by a force F. In that case, F would equal $T_{BC}$, by the definition of the tension (and this would be true whether the system were accelerating or not). As mentioned above though, the tension forces at the two ends of BC would only be equal (when there is acceleration) if the string has negligible mass.

(2.3) Consider the situation where string AB (with original length $l_1$ and modulus of elasticity $\lambda_1$) is horizontal, with A fixed, and joined to the string BC (with original length $l_2$ and modulus of elasticity $\lambda_2$), which is also horizontal, with C fixed. Suppose that the distance AC is $l_1 + l_2 + x$. Let the extensions of AB and BC be $x_1$ and $x_2$, so that $x_1 + x_2 = x$.




Then $T_{AB} = T_{BC}$, as before, and $T_{AB} = \frac{\lambda_1 x_1}{l_1}$ and $T_{BC} = \frac{\lambda_2 x_2}{l_2}$

Therefore $\frac{\lambda_1 x_1}{l_1} = \frac{\lambda_2 x_2}{l_2} = \frac{\lambda_2 (x - x_1)}{l_2}$,

so that $x_1(\lambda_1 l_2 + \lambda_2 l_1) = \lambda_2 l_1 x$

and thus $x_1 = \frac{\lambda_2 l_1 x}{\lambda_1 l_2 + \lambda_2 l_1}$

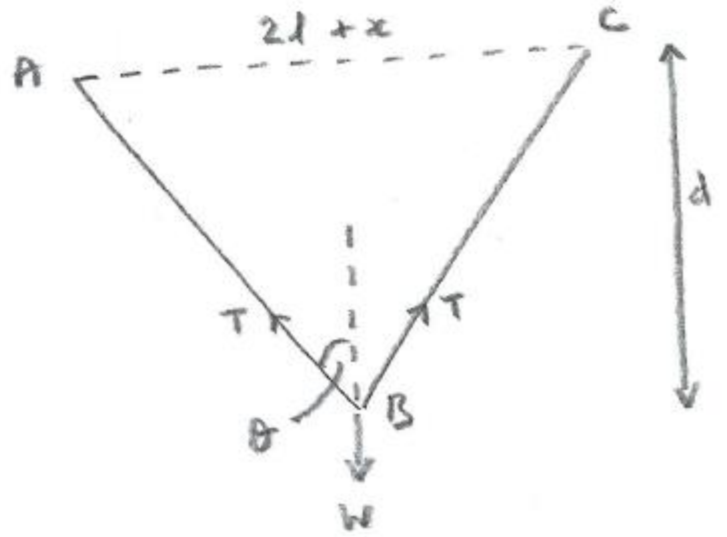
(2.4) Suppose that a load of weight $W$ is now applied at B, such that, in an equilibrium position, B is at a distance $d$ below its original level (though not necessarily directly below its original position).

**Special Case**: $l_1 = l_2 = l$ and $\lambda_1 = \lambda_2 = \lambda$, and also $x_1 = x_2 = \frac{1}{2}x$;

ie the strings AB and BC are identical, and B is directly below the mid-point of AC.

By symmetry, the new tensions in the strings are equal; say $T$.

The force diagram for the load at B is shown below.




Resolving vertically, $2Tcos\theta = W$ (1)

Also, $tan\theta = \frac{l+\frac{x}{2}}{d}$ (2) and, if $y$ is the new extension of AB,

$l + y = \sqrt{(l + \frac{x}{2})^2 + d^2}$ (3) and $T = \frac{\lambda y}{l}$ (4)

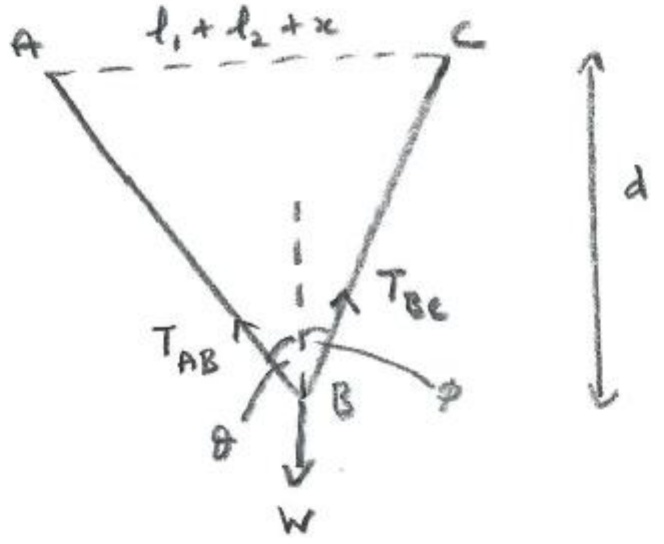
Suppose that $\lambda = 10, l = 2, x = 2, d = 4$. Then we can find $W$ as follows:

From (2), $tan\theta = \frac{3}{4}$, so that $cos\theta = \frac{4}{5}$;

so that (3) $\Rightarrow 2 + y = 5$, and (4) $\Rightarrow T = \frac{30}{2} = 15$

Then, from (1), $W = 30\left(\frac{4}{5}\right) = 24$

**General case**




Suppose that $l_1 = 1, l_2 = 2, \lambda_1 = 8, \lambda_2 = 12, x = 2, d = 4$.

Let $y_1$ and $y_2$ be the new extensions of AB and BC respectively.

Here we can't take advantage of symmetry. Equations can be set up (as shown below), but they would be difficult to solve.

Hence exam questions are likely to avoid this situation, and be based instead on the special case, where the load hangs below the mid-point.

The equations are:

Resolving vertically, $T_{AB}cos\theta + T_{BC}cos\phi = W$ (1)

Resolving horizontally, $T_{AB}sin\theta = T_{BC}sin\phi$ (2)

And $T_{AB} = \frac{\lambda_1 y_1}{l_1} = 8y_1$ (3a) and $T_{BC} = \frac{\lambda_2 y_2}{l_2} = 6y_2$ (3a)

Then $cos\theta = \frac{d}{l_1+y_1} = \frac{4}{1+y_1}$ (4a) and $cos\phi = \frac{d}{l_2+y_2} = \frac{4}{2+y_2}$ (4b)

Also, $dtan\theta + dtan\phi = l_1 + l_2 + x$,

so that $tan\theta + tan\phi = \frac{5}{4}$ (5)

(Allowing for the known values, there are 7 unknowns and 7 equations.)

**(3) Multiple springs**

(a) Springs in series

Consider two springs of stiffness $k_1$&$k_2$ , held in equilibrium in series, as shown in Figure 1.


Figure 1

Draw separate force diagrams for the two springs, as in Figure 2.

Figure 2

By N2L, $T_1 = F$ and $T_2 = F$

(Also, $T_1 = T_2$ , by N3L. )

By Hooke's Law, $F = k_1 e_1$ & $F = k_2 e_2$,

where $e_1$&$e_2$ are the extensions of the two springs.

Let the stiffness of the combined springs be $k$.

Then $F = k(e_1 + e_2)$

and so $k = \frac{F}{e_1+e_2} = \frac{F}{\frac{F}{k_1}+\frac{F}{k_2}}$

$\Rightarrow \frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$

Note: This can be extended to more than two springs, by replacing $\frac{1}{k_2}$ with $\frac{1}{k_3} + \frac{1}{k_4}$ , and then re-labelling to give $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$

(b) Springs in parallel

Consider two springs of the same original length and stiffnesses $k_1$&$k_2$ , held in equilibrium in parallel, as shown in Figure 3.

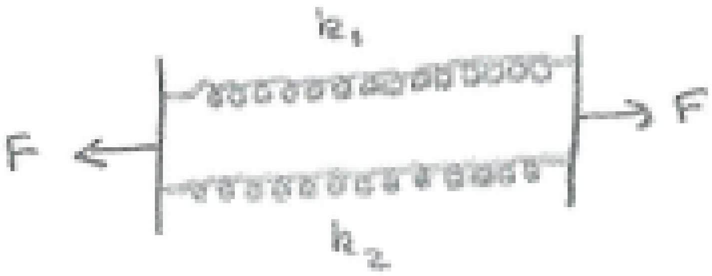



Figure 3

(Note: This system only makes sense if the original lengths are the same, so that when no force is applied, the springs both reach to the two sides.)

The left-hand side of this system is equivalent to that shown in Figure 4, with forces being applied to a light bar (akin to a towbar between a car and a trailer).

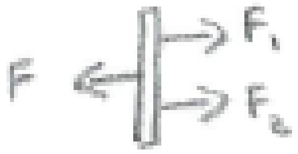



Figure 4

Let the stiffness of the combined springs be $k$, with extension e.

As the springs have the same original length, their extensions are both equal to e.

Then $F = ke$ , $F_1 = k_1 e$ , $F_2 = k_2 e$, and $F = F_1 + F_2$

Hence $ke = k_1 e + k_2 e$ , and so $k = k_1 + k_2$