Groups - Exercises (Sol'ns) (8 pages; 16/8/19)

(1) Multiplication modulo m (or just mod m), denoted by $\times_m$, is defined on the set $\{0,1,2, \ldots , m-1\}$ by carrying out ordinary multiplication and taking the remainder when the product is divided by m. For example, $5 \times_3 4 = 2$.

Show that $\times_5$ is a closed and commutative binary operation on the set $\{0,1,2, \ldots ,4\}$, and identify the inverse of each element, where it exists.

Solution

(i) The Cayley table is:

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

As $\times_5$ is defined for all pairs of elements of the set, it is a binary operation.

Each of the products in the Cayley table is an element of the set, and so the operation is closed.

It is commuative because of symmetry about the leading diagonal.

[Note: Associativity is harder to establish, but can usually just be asserted in the case of modular addition or multiplication.]

The element 1 leaves all elements unchanged, and is therefore the identity element.

The inverses are as follows:

0: doesn't exist
1: 1
2: 3
3: 2
4: 4

(2)(i) Show that the set {1,4,7,13} forms a group, under multiplication modulo 15.

(ii) Find the generators of the group.

(iii) Establish whether the group is cyclic.

(iv) Identify all the subgroups.

Solution

(i) The Cayley table is:

	1	4	7	13
1	1	4	7	13
4	4	1	13	7
7	7	13	4	1
13	13	7	1	4

The operation is closed.

There is an identity element (1).

The identity element appears in each row, so that each element has an inverse.

Associativity follows from the associativity of ordinary multiplication.

So the conditions for a group are satisfied.

(ii) $4^2 \equiv 1$

$7^2 \equiv 4; 7^3 = 4 \times 7 = 28 \equiv 13; 7^4 = 13 \times 7 = 91 \equiv 1$

$13^2 \equiv 4; 13^3 = 4 \times 13 = 52 \equiv 7; 13^4 = 7 \times 13 = 91 \equiv 1$

So 7 and 13 are the generators of the group.

(iii) The elements of the group can be written as:

$1, 7, 7^2, 7^3$ (for example), and so the group is cyclic.

[Also, the elements have periods 1,2,4,4, which is the pattern for a cyclic group of order 4.]

(iv) The subgroups are $\{1\}, \{1,4\}, \{1,7,4,13\}$

(3) For the group $\left\{x, 1-x, \frac{1}{x}, \frac{1}{1-x}, \frac{x-1}{x}, \ \frac{x}{x-1}\right\}$ under composition of functions, where $x \in \mathbb{R}, x \neq 0,1$:

(i) Establish whether the group is abelian.

(ii) Find the periods of the elements of the group, and hence identify its proper subgroups.

Solution

(i) Let $e = x, a = 1 - x, b = \frac{1}{x}, c = \frac{1}{1-x}, d = \frac{x-1}{x}, f = \frac{x}{x-1}$

The Cayley table is:

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d	f	b	c
b	b	c	e	a	f	d

c	c	b	f	d	e	a
d	d	f	a	e	c	b
f	f	d	c	b	a	e

For example, $ab = d$ but $ba = c$, so $ab \neq ba$, and hence the group is not abelian.

(ii) $a^2 = e$, so a is of order 2

$b^2 = e$, so b is of order 2

$c^2 = d$, $c^3 = c(c^2) = cd = e$ so c is of order 3

[note that $c(c^2) = c(c)(c) = (c^2)c$, by associativity, so that

$cd = dc$ (even though the group isn't abelian)]

$d^2 = c, d^3 = d(d^2) = dc = e$ so d is of order 3

$f^2 = e$, so f is of order 2

Hence the proper subgroups are: $\{e, a\}, \{e, b\}, \{e, f\}, \{e, c, d\}$

(4) Establish which of the following groups are isomorphic to each other:

(i) {0,1,2,3} ; addition modulo 4

(ii) {1,2,4,8} ; multiplication modulo 15

(iii) {3,6,9,12} ; multiplication modulo 15

(iv) {1,3,5,7} ; multiplication modulo 8

(v) $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\right\}$; matrix multiplication

(vi) $\{1, i, -1, -i\}$; multiplication of complex numbers

Solution

Groups of order 4 are either cyclic or Klein 4, as established by their Cayley tables (a cyclic group will have elements of period 1,2,4,4; a Klein 4 group will have elements of period 1,2,2,2).

(i) {0,1,2,3} ; addition modulo 4

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Cyclic group (0 has period 1, 1 has period 4, 2 has period 2, 3 has period 4).

(ii) {1,2,4,8} ; multiplication modulo 15

	1	2	4	8
1	1	2	4	8
2	2	4	8	1
4	4	8	1	2

8	8	1	2	4

Cyclic group (1 has period 1, 2 has period 4, 4 has period 2, 8 has period 4).

(iii) {3,6,9,12} ; multiplication modulo 15

	3	6	9	12
3	9	3	12	6
6	3	6	9	12
9	12	9	6	3
12	6	12	3	9

Cyclic group (3 has period 4, 6 has period 1, 9 has period 2, 12 has period 4).

(iv) {1,3,5,7} ; multiplication modulo 8

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Klein 4-group (1 has period 1, 3 has period 2, 5 has period 2, 7 has period 2).

(v) $\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\}$; matrix multiplication

[$a = \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$: 180° rotation

$b = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$: reflection in x-axis

$c = \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$: reflection in y-axis]

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Klein 4-group (e has period 1, a has period 2, b has period 2, c has period 2).

(vi) $\{1, i, -1, -i\}$; multiplication of complex numbers

	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

Cyclic group (1 has period 1, i has period 4, -1 has period 2, $-i$ has period 4).