Groups - Exercises (2 pages; 16/8/19)

(1) Multiplication modulo m (or just mod m), denoted by $\times_m$, is defined on the set $\{0,1,2, \dots , m-1\}$ by carrying out ordinary multiplication and taking the remainder when the product is divided by m. For example, $5 \times_3 4 = 2$.

Show that $\times_5$ is a closed and commutative binary operation on the set $\{0,1,2, \dots ,4\}$, and identify the inverse of each element, where it exists.

(2)(i) Show that the set $\{1,4,7,13\}$ forms a group, under multiplication modulo 15.

(ii) Find the generators of the group.

(iii) Establish whether the group is cyclic.

(iv) Identify all the subgroups.

(3) For the group $\left\{x, 1-x, \frac{1}{x}, \frac{1}{1-x}, \frac{x-1}{x}, \ \frac{x}{x-1}\right\}$ under composition of functions, where $x \in \mathbb{R}, x \neq 0,1$:

(i) Establish whether the group is abelian.

(ii) Find the periods of the elements of the group, and hence identify its proper subgroups.

(4) Establish which of the following groups are isomorphic to each other:

(i) $\{0,1,2,3\}$; addition modulo 4

(ii) $\{1,2,4,8\}$; multiplication modulo 15

(iii) $\{3,6,9,12\}$; multiplication modulo 15

(iv) $\{1,3,5,7\}$; multiplication modulo 8

(v) $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix},\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix},\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\right\}$; matrix multiplication

(vi) $\{1, i, -1, -i\}$; multiplication of complex numbers