Gravitational Potential Energy (3 pages; 15/2/19)

Reference: "Classical Mechanics" (French and Ebison)

[It is assumed, unless stated otherwise, that the Earth's gravity is the only force acting on a given particle.]

Particle outside the Earth

According to Newton's Law of Gravitation, the force on a particle of mass m at a distance r from the centre of the Earth is $\mathrm{F} = \frac{GMm}{r^2}$, where G is the Gravitational constant (6.670×10^{-11} N$m^2 kg^{-2}$) and M is the mass of the Earth [assuming that $r \geq R$, the radius of the Earth].

The work done by gravity in moving a particle from $x = \infty$ to $x = r$, where x is the distance from the centre of the Earth, is

$\int_{\infty}^{r} -\frac{GMm}{x^2}\, dx$ (the force is negative because it acts in the opposite direction to increasing x)

$= [\frac{GMm}{x}]_{\infty}^{r} = \frac{GMm}{r}$

This work is equal to the increase in KE of the particle (by the Work-Energy principle).

If we take the total energy of a particle at infinity to be zero, with zero KE (and therefore zero PE), the KE at $x = r$ is therefore $\frac{GMm}{r}$, and since

KE + PE = 0, it follows that, when $x = r$, PE $= -\frac{GMm}{r}$ [provided $r \geq R$]

Notes

(i) The PE increases with distance from the surface of the Earth: from

$-\frac{GMm}{R}$ to 0

(ii) The usual expression for PE of mgh (where h is the distance above the ground) can be arrived at by adding the constant $\frac{GMm}{R}$ to the expression

$-\frac{GMm}{r}$, so that the surface of the Earth becomes the zero of PE (ie when

$r = R$).

The PE then becomes $\frac{GMm}{R} - \frac{GMm}{r} = \frac{GMm}{R^2}(R - \frac{R^2}{r})$

g is then taken to be $\frac{GM}{R^2}$ [see Note (iii)] to give:

$mg.\frac{Rr - R^2}{r}$ and if we write $h = r - R$ then we have:

$mg.\frac{(R(R+h) - R^2}{R+h} = mg.\frac{Rh}{R+h}$ or $mgh.\frac{R}{R+h}$

which is close to mgh if h is small

(iii) With $G = 6.670 \text{ x } 10^{-11}\ \text{N}m^2kg^{-2}$, $M = 5.972 \times 10^{24}kg$ and

$R = 6371\ km = 6.371 \times 10^6 m,\ \ g = \frac{GM}{R^2} = 0.981 \times 10^1 = 9.81ms^{-2}$

Particle inside the Earth

The form of the gravitational force, F as $\frac{GMm}{r^2}$ only applies if $r \geq R$, which is just as well, as otherwise it would be infinite at the centre of the Earth.

It can be shown that, for $r < R$, F $= \frac{GMm}{R^3}\ r$ - although this only holds for a very simplified model of the Earth.

This is demonstrated by first of all showing that the force due to a thin spherical shell of mass M and radius a is $\frac{GMm}{r^2}$ if $r \geq a$ (ie as though all of the mass were concentrated at the centre), but zero if $r < a$ (the force between particles follows an inverse square law, and it turns out that, for a particle near the edge of the shell, the greater mass of the other side is balanced by its greater distance).

Then, when the contributions from multiple shells are added up, only those with radius <r contribute to the force (so that F approaches zero as r approaches zero).

If a sphere is homogeneous in its density (which unfortunately isn't true for the Earth), then the net effect on a particle at distance r from its centre is equivalent to that caused by a particle of mass $M\left(\frac{r}{R}\right)^3$ at the centre, since only shells within distance r of the centre contribute to the force, and the linear proportion $\frac{r}{R}$ gives rise to a proportion by volume of $\left(\frac{r}{R}\right)^3$.

This force from the equivalent particle is $\frac{G\left(M.\frac{r^3}{R^3}\right)m}{r^2} = \frac{GMm}{R^3}r$.

Then, when $x = r(< R), PE = -KE$ created by gravity in bringing the particle in from infinity (as the particle started with zero total energy)

$= -$ Work done by gravity in bringing the particle in from infinity

$= -\{ \int_\infty^R -\frac{GMm}{x^2}dx + \int_R^r -\frac{GMm}{R^3}x\, dx \}$

[noting that, as before, the gravitational force is in the negative x direction, and so does $-\frac{GMm}{x^2}\ dx$ work in the positive x direction, for the infinitesimal displacement dx (though strictly speaking, we should use the symbol δx, which becomes dx, in the limit as the summation becomes an integral)]

$= -\{ [\frac{GMm}{x}]_\infty^R + [-\frac{GMmx^2}{2R^3}]_R^r \}$

$= -\frac{GMm}{R} + \frac{GMm}{2R^3}(r^2 - R^2)$

$= -\frac{GMm}{2R^3}(2R^2 - r^2 + R^2)$

$= -\frac{GMm}{2R^3}(3R^2 - r^2)$

And when $r = 0$,

$PE = -\frac{3GMm}{2R}$

Notes

(i) The result for the thin shell is valid provided that the density is equal for equal distances from the centre.

(ii) $\frac{GMm}{R^3}r$ becomes $\frac{GMm}{r^2}$ when $r = R$, so that there is no discontinuity.