Geometry – Q4 [Problem/M] (24/5/21)

Show that the shortest distance from the line $ax + by = c$ to the Origin is $\frac{c}{\sqrt{a^2+b^2}}$, for the case where the line has a positive gradient, and a positive y-intercept.

[This is analogous to the shortest distance from the plane $n_1x + n_2y + n_3z = d$ to the Origin; namely $\frac{d}{\sqrt{{n_1}^2+{n_2}^2+{n_3}^2}}$]

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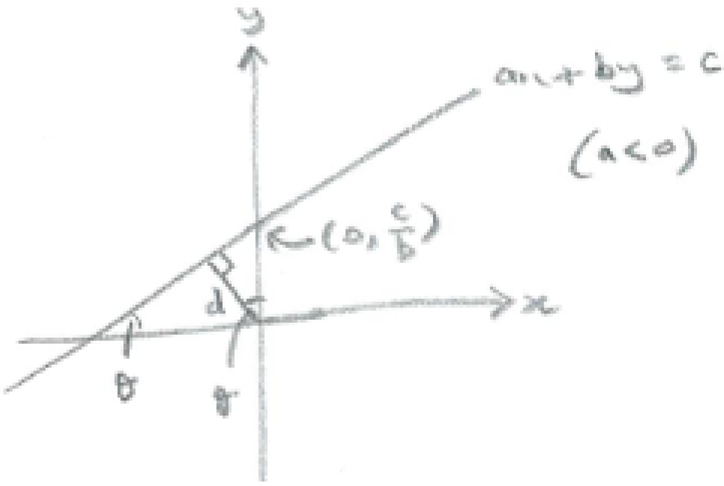
Solution

Method 1 [using $tan\theta$]



Referring to the diagram, $ax + by = c \Rightarrow y = \frac{c}{b} - \frac{a}{b}x$,

so that $tan\theta = -\frac{a}{b}$

and $d = \frac{c}{b}\ cos\theta = \frac{c}{b}.\frac{1}{\sqrt{tan^2\theta+1}} = \frac{c}{b}\ .\frac{1}{\sqrt{\frac{a^2}{b^2}+1}} = \frac{c}{\sqrt{a^2+b^2}}$

Method 2 [vectors: shortest distance from a point to a line]

The line has vector eq'n $\underline{r} = \begin{pmatrix} 0 \\ \frac{c}{b} \end{pmatrix} + \lambda \begin{pmatrix} b \\ -a \end{pmatrix}$ (as the gradient is $-\frac{a}{b}$)

Let the point closest to the Origin (P, say) have parameter $\lambda = \lambda_P$

Then $\overrightarrow{OP}.\begin{pmatrix} b \\ -a \end{pmatrix} = 0$

so that $\begin{pmatrix} \lambda_P b \\ \frac{c}{b} - \lambda_P a \end{pmatrix}.\begin{pmatrix} b \\ -a \end{pmatrix} = 0,$

and $b^2\lambda_P - \frac{ac}{b} + a^2\lambda_P = 0,$

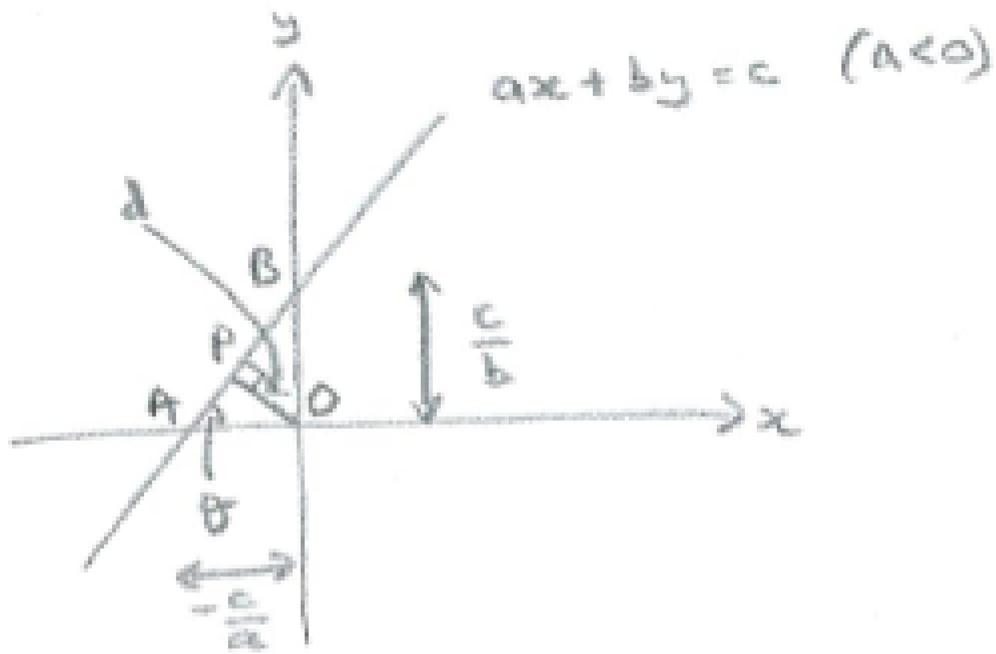
giving $\lambda_P = \frac{ac}{b(a^2+b^2)}$

Then $\overrightarrow{OP} = \frac{1}{(a^2+b^2)}\begin{pmatrix} ac \\ \frac{c}{b}(a^2 + b^2) - \frac{a^2c}{b} \end{pmatrix}$

$= \frac{1}{(a^2+b^2)}\begin{pmatrix} ac \\ bc \end{pmatrix}$

and $\left|\overrightarrow{OP}\right| = \frac{c}{(a^2+b^2)}\left|\begin{matrix} a \\ b \end{matrix}\right| = \frac{c}{(a^2+b^2)}\sqrt{a^2 + b^2} = \frac{c}{\sqrt{a^2+b^2}}$

Method 3 (similar triangles)



OPA and AOB are similar triangles

So $\frac{-\left(\frac{c}{a}\right)}{d} = \frac{AB}{\left(\frac{c}{b}\right)}$, and $AB^2 = \left(\frac{c}{a}\right)^2 + \left(\frac{c}{b}\right)^2$

$$\Rightarrow d = \frac{-\left(\frac{c}{a}\right)\left(\frac{c}{b}\right)}{AB} = \frac{-c^2}{ab\sqrt{\left(\frac{c}{a}\right)^2+\left(\frac{c}{b}\right)^2}} = \frac{c}{\sqrt{b^2+a^2}}$$

(noting that, as $a < 0$, $a = -\sqrt{a^2}$)

Method 4 (Intersection of perpendicular lines)

Let L be the line $ax + by = c$, and L' the line perpendicular to L.

The gradient of L is $-\frac{a}{b}$, and so the gradient of L' is $\frac{b}{a}$.

Hence the eq'n of L' is $y = \frac{b}{a}x$

Let the intersection of L & L' be (x_P, y_P).

Then $ax_P + b(\frac{b}{a}x_P) = c$

$\Rightarrow x_P(a^2 + b^2) = ac$

Then $d^2 = {x_P}^2 + {y_P}^2 = {x_P}^2(1 + \left(\frac{b}{a}\right)^2)$,

so that $d = (-x_P)\sqrt{\frac{a^2+b^2}{a^2}}$ $\quad (a < 0, c > 0 \Rightarrow x_P < 0)$

$= -\frac{ac}{a^2+b^2}\sqrt{\frac{a^2+b^2}{a^2}}$

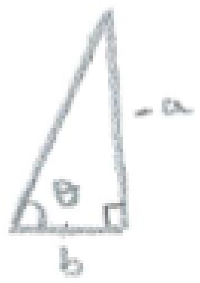
$= \frac{c}{\sqrt{a^2+b^2}}$ $\;(a < 0$, so that $a = -\sqrt{a^2}$)

Method 5 (trigonometry)

Referring to the diagram for Method 3,

$tan\theta = \frac{\left(\frac{c}{b}\right)}{\left(-\frac{c}{a}\right)} = -\frac{a}{b}$ and $sin\theta = \frac{d}{\left(-\frac{c}{a}\right)} = -\frac{ad}{c}$

Creating a right-angled triangle where $tan\theta = -\frac{a}{b}$, as below, shows that we can also write $sin\theta = \frac{-a}{\sqrt{a^2+b^2}}$



Equating the two expressions for $sin\theta$ then gives

$-\frac{ad}{c} = \frac{-a}{\sqrt{a^2+b^2}}$,

so that $d = \frac{c}{\sqrt{a^2+b^2}}$

Method 6 (stationary point)

Consider a general point Q on the line $ax + by = c$ with coordinates (x, y)

Then d will be the minimum value of OQ as x varies,

and $OQ^2 = x^2 + \left(\frac{c-ax}{b}\right)^2$ (1)

This occurs when $\frac{d}{dx}(OQ^2) = 0$;

ie when $2x + 2\left(\frac{c-ax}{b}\right)\left(-\frac{a}{b}\right) = 0$

$\Rightarrow xb^2 - ca + a^2x = 0$

$\Rightarrow x = \frac{ca}{a^2+b^2}$

Then, from (1), $OQ^2 = x^2 + \left(\frac{c-ax}{b}\right)^2$

$$= \left(\frac{ca}{a^2+b^2}\right)^2 + \left(\frac{c\left(a^2+b^2\right)-ca^2}{b(a^2+b^2)}\right)^2$$

$$= \left(\frac{ca}{a^2+b^2}\right)^2 + \left(\frac{cb}{a^2+b^2}\right)^2$$

$$= \frac{c^2(a^2+b^2)}{(a^2+b^2)^2}$$

and $d = OQ = \frac{c}{\sqrt{a^2+b^2}}$