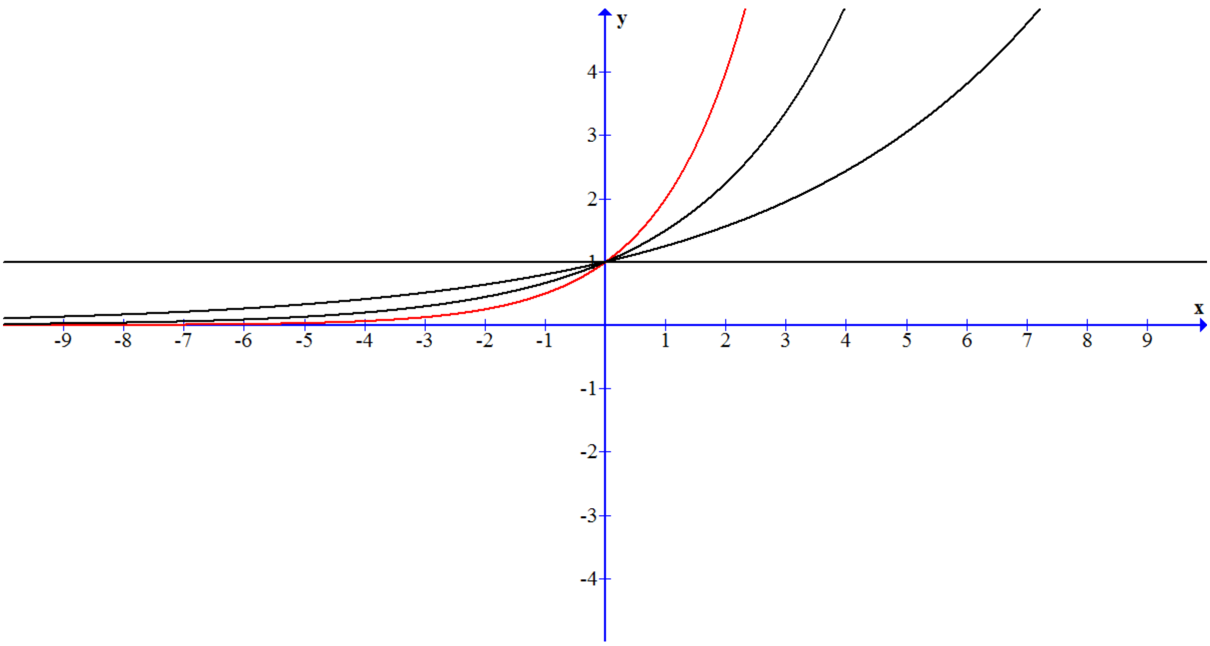
Functions - Miscellaneous (7 pages; 4/9/18)

Contents

(A) Exponential Functions

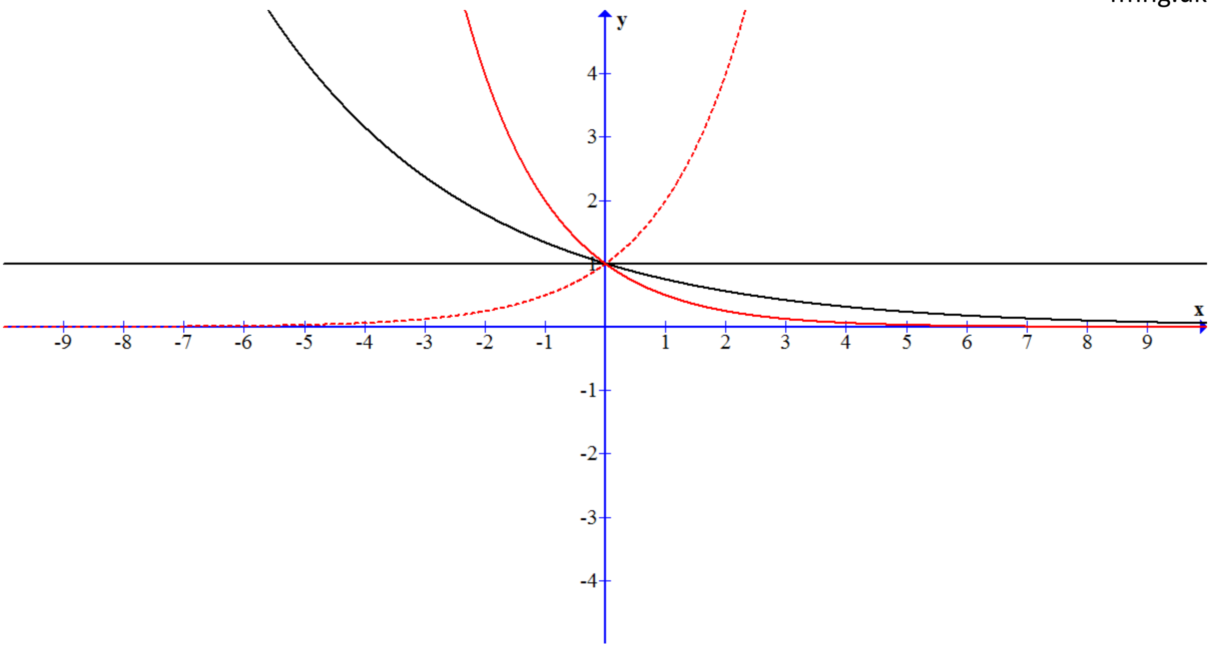
Connection between 2^x and 2^{-x}

(i) Graphs of $y = 2^x$ (red), $y = 1.5^x$, $y = 1.25^x$ and $y = 1^x$:



(ii) The pattern continues for numbers below 1.

Graphs of $y = 2^x$ (dotted red), $y = 1^x$, $y = 0.75^x$ and $y = 0.5^x$ (red):

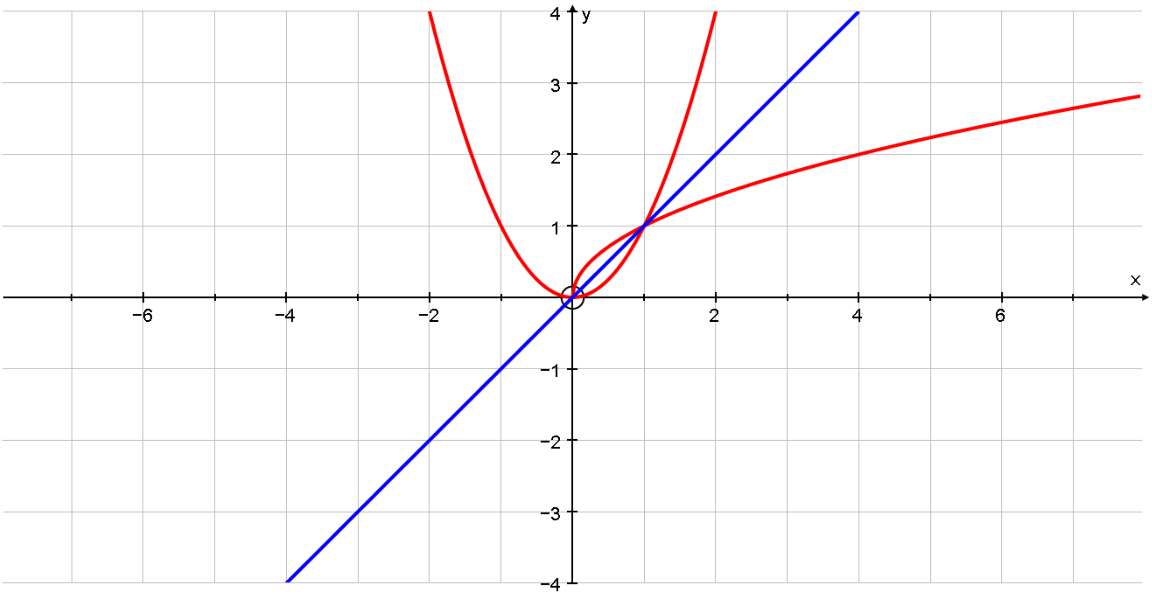


(iii) $0.5^x = \left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$

and $y = 2^{-x}$ is the reflection of $y = 2^x$ in the y-axis

(B) Sketching inverses

Example: $y = x^2$



There are two possible approaches:

(a) Obtain the inverse function algebraically:

(i) Make x the subject of the equation, so that x = $\sqrt{y}$

(only the +ve root is used, in order for the inverse to be a function)

(Whereas the original function represented a mapping from x to y, the new function represents the opposite (or 'inverse') mapping from y to x.)

(ii) Swap the roles of x and y (in order for the new function to have x values on the horizontal axis), to give y = $\sqrt{x}$

or (b) Reflect the function y = x^2 in the line y = x

The point (a, a^2) then moves to (a^2, a), to obtain the inverse mapping:

the y coordinate is now the square root of the x coordinate.

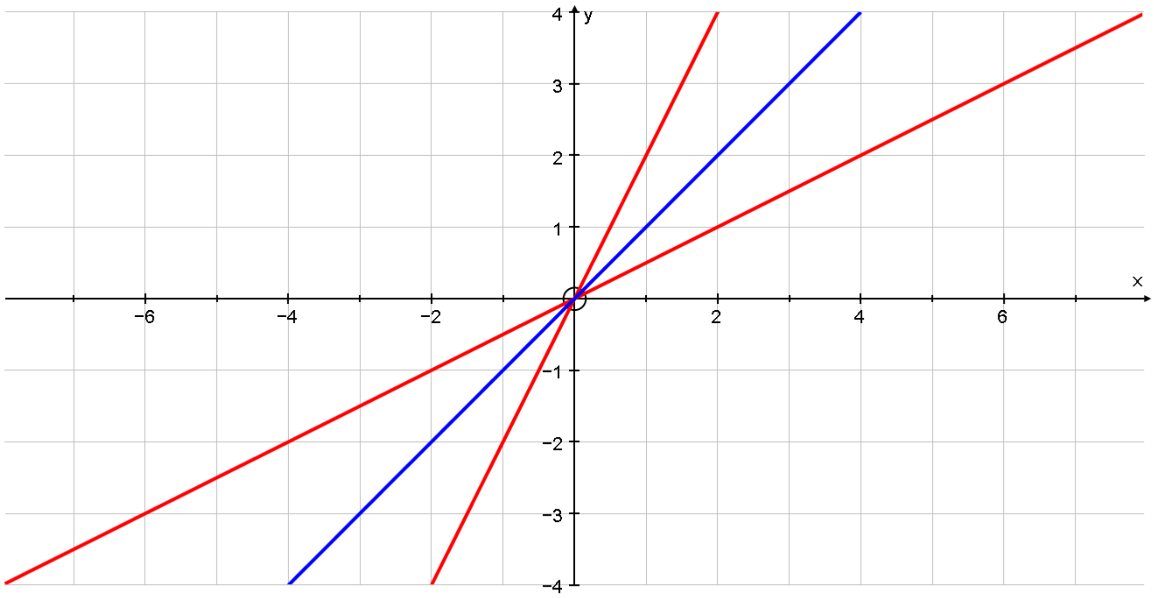
Alternatively, the equivalent transformation of reflecting in the y axis (producing no change in this case) and then rotating clockwise through 90° may be easier to visualise.

Note that f(x) and $f^{-1}(x)$ should cross on the line y = x (if they intersect).

Also, ensure that the x and y axes have the same scale; otherwise there won't be any symmetry.

(C) Differentiating an inverse function

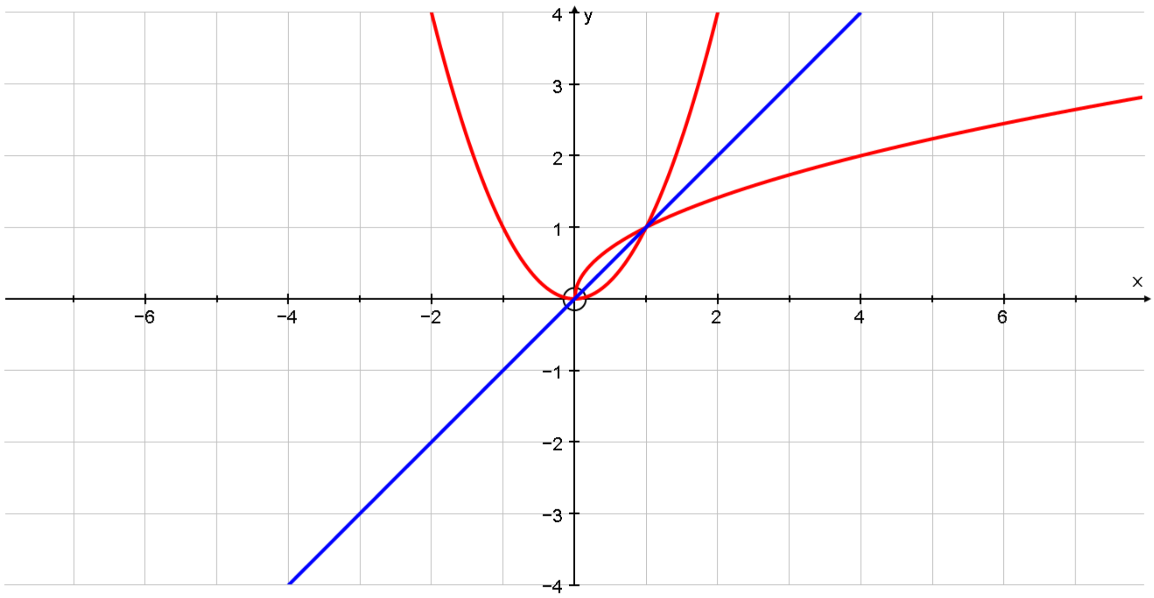
Consider first of all the simple example y = 2x:



The inverse function is y = x/2

The gradient of the inverse function (1/2) is the reciprocal of the gradient of the original function (2); though in this example it is a constant gradient.

For a more complicated function such as $y = x^2$, the gradient of the inverse function $y = \sqrt{x}$ at the point (b,a) will equal the reciprocal of the gradient of $y = x^2$ at the point (a,b) - since (a,b) is the reflection of (b,a) in the line y = x.



In general, $\frac{d}{dx} f^{-1}(x) \, |x = b \quad = \frac{1}{\frac{d}{dx} f(x)|x=a}$

where f(a) = b, so that $f^{-1}(b) = a$

[Note: "|x = b" means "when x = b"]

This is also written as $\frac{dx}{dy} \, |y=b = \frac{1}{\frac{dy}{dx}} \, |x = a$

(E) Definitions

(1) Mappings and functions; injections

Consider the following examples of mappings, where $x \in \mathbb{R}$

(a) $f(x) = 2x$

This is a $1 - 1$ mapping. The lefthand 1 refers to the fact that eg only 1 value of x maps to 4 (namely 2), whilst the righthand 1 refers to the fact that eg there is only one image of 3 (namely 6).

Any mapping with a righthand 1 is described as a function.

(b) $g(x) = x^2$

This is a $many - 1$ mapping. The lefthand 'many' refers to the fact that eg there is more than 1 value of x that maps to 9 (namely 3 & -3). As for (a), there is eg only one image of 4 (namely 16); hence the righthand 1.

The presence of the righthand 1 makes $g(x)$ a function.

(c) $h(x) = \pm\sqrt{x}$

This is a $1 - many$ mapping. The lefthand 1 refers to the fact that eg there is only 1 value of x that maps to 5 (namely 25). As eg there is more than 1 image of 49 (namely $7 \ \& -7$), we have a righthand 'many', and $h(x)$ is therefore not a function.

(d) $j(x) = \pm\sqrt{25 - x^2}$ (a circle)

This is a $many - many$ mapping. The lefthand 'many' refers to the fact that eg there is more than 1 value of x that maps to 3 (namely $4 \ \& -4$). And as eg there is more than 1 image of 0 (namely $5 \ \& -5$), we have a righthand 'many', and $j(x)$ is therefore not a function.

An injection is a $1 - 1$ function (as opposed to a $many - 1$ function). (A function can also be described as injective.)

(2) Domain, codomain, range; surjections

Suppose that a function is defined in the following way:

$f: \mathbb{Z} \to \mathbb{Z}; f(x) = 2x$ [ie specifying that $x \in \mathbb{Z}$ & $f(x) \in \mathbb{Z}$],

then the domain of f is defined as the set of allowable values for x; ie $\mathbb{Z}$ in this case.

The codomain is defined as the set of allowable values for $f(x)$;

ie also $\mathbb{Z}$ in this case [note that this is before we have investigated whether the nature of the function places any constraints on the possible values of $f(x)$];

The range of f is defined as the set of values that $f(x)$ will actually take; ie in this case allowing for the fact that $2x$ has to be even; so the range here is the set of even integers.

Where the codomain is $\mathbb{R}$ (or a subset of $\mathbb{R}$), then the range is usually the same as the codomain. Many textbooks and exam boards ignore the codomain, and just work with the range.

A function for which the range is the whole of the codomain is described as a surjection. (A function can also be described as surjective.)

(3) A bijection is a function that is both injective and surjective. (A function can also be described as bijective.)

In other words, a $1-1$ function where the range is the same as the codomain. For example, $f: \mathbb{R} \to \mathbb{R}; f(x) = 2x$