

Numerical Solution of Equations - Fixed Point Iteration

(15 pages; 22/10/18)

(1) Suppose that we wish to solve the equation $x = g(x)$ approximately.

Let our first estimate be x_0 , and let $x_1 = g(x_0)$, $x_2 = g(x_1)$ etc.

If it happens that the sequence of x_r converges on a particular value α , then $\alpha = g(\alpha)$ and α will be a solution of the equation.

(2) If we wish to solve the equation $x^3 - x = 1$ numerically, it can be rearranged into the following forms (for example):

(A) $x = x^3 - 1$

(B) $x = \sqrt[3]{x + 1}$

(C) $x = \frac{1}{x} + \frac{1}{x^2}$

(by dividing the original equation by x^2 to give $x - \frac{1}{x} = \frac{1}{x^2}$)

(3) The iterations can be carried out by calculator.

We need to find a suitable starting point x_0 . To do this we can employ the Change of Sign method, for example; demonstrating (with a bit of trial and error) that for $f(x) = x^3 - x - 1$, $f(1) < 0$ and $f(2) > 0$, so that a solution lies between 1 and 2. Note that other solutions may exist elsewhere.

So let $x_0 = 1$, say.

For most Casio models, type in the following for (B):

1 = ANS DEL

$$(ANS + 1)^{1/3}$$

= [repeatedly]

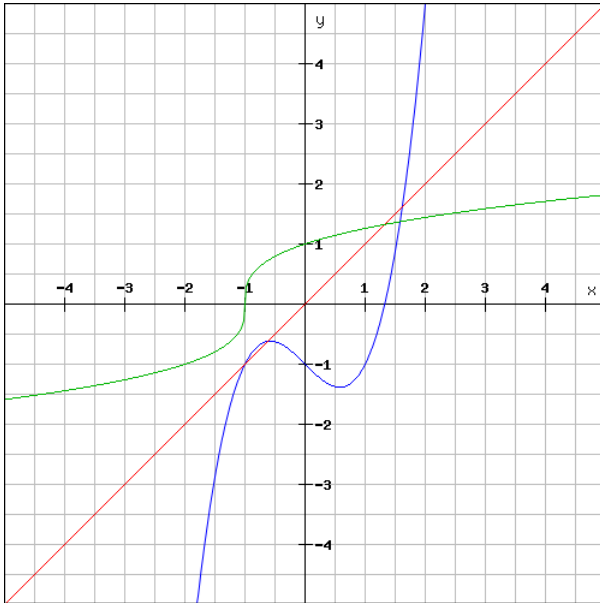
The iterations are as follows (to 5dp), with the results shown for (A) and (C) as well:

(A)	(B)	(C)
1	1	1
0	1.25992	2
-1	1.31229	0.75
-2	1.32235	3.11111
-9	1.32427	0.42475
-730	1.32463	7.89734
	1.32470	0.14266
	1.32472	56.14607
	1.32472	

Thus only (B) converges.

(4) The diagram below shows the graphs of $y = x^3 - x - 1$ (crossing the x -axis at 1.32472), $y = \sqrt[3]{x+1}$ (in green)

and $y = x$.

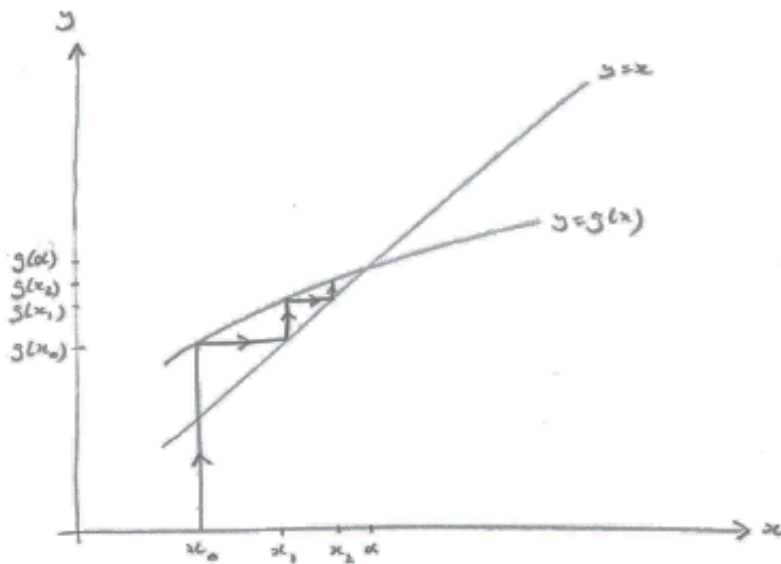


Note that the graphs of $y = \sqrt[3]{x+1}$ and $y = x$ intersect at $x = 1.32472$, corresponding to the solution of $x = g(x)$ for (B).

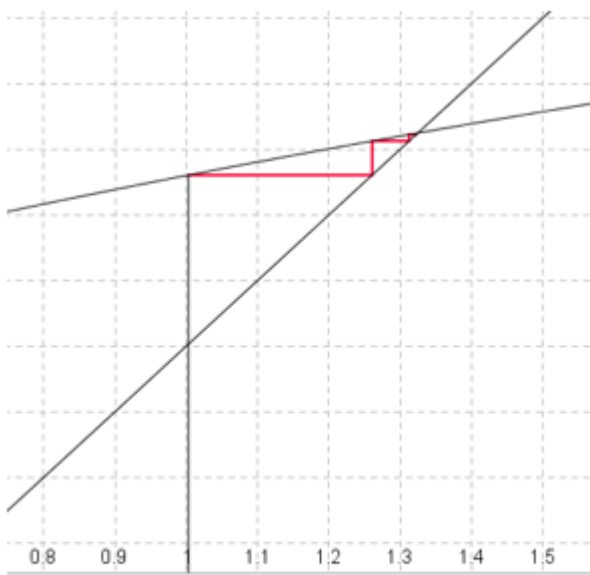
(5) The diagram below shows how the iterations for (B) approach $\alpha = 1.32472$

Starting with $x = x_0$, we obtain $x_1 = g(x_0)$ by following the vertical line up to the curve $y = g(x)$; x_1 is then the y -coordinate of the point reached; this is then turned into a point on the x -axis by following the horizontal line along to the line $y = x$; the

x -coordinate of the point reached is $x_1 = g(x_0)$; the process is then repeated to find x_2 etc (though the vertical lines don't need to start on the x -axis).



For this particular series of iterations, the x_r approach α from below in a 'staircase' fashion.



$$g(x) = \sqrt[3]{x+1}$$

(6) We will now show that the convergence (or otherwise) of iterations of $x = g(x)$ depends on the value of $g'(\alpha)$.

From the diagram in (5), provided that x_r is reasonably close to α ,

$$g'(\alpha) \approx \frac{g(\alpha) - g(x_r)}{\alpha - x_r} = \frac{\alpha - g(x_r)}{\alpha - x_r} = \frac{g(x_r) - \alpha}{x_r - \alpha} \quad (1),$$

using the fact that α is a solution of $x = g(x)$, so that $\alpha = g(\alpha)$.

Let $e_r = x_r - \alpha$ be the 'error' associated with x_r .

[Note: some textbooks define e_r as $\alpha - x_r$]

Then $e_{r+1} = x_{r+1} - \alpha = g(x_r) - \alpha \approx g'(\alpha)(x_r - \alpha)$, from (1)

Thus $e_{r+1} \approx g'(\alpha) \cdot e_r$

This is an example of what is termed **1st order convergence**,

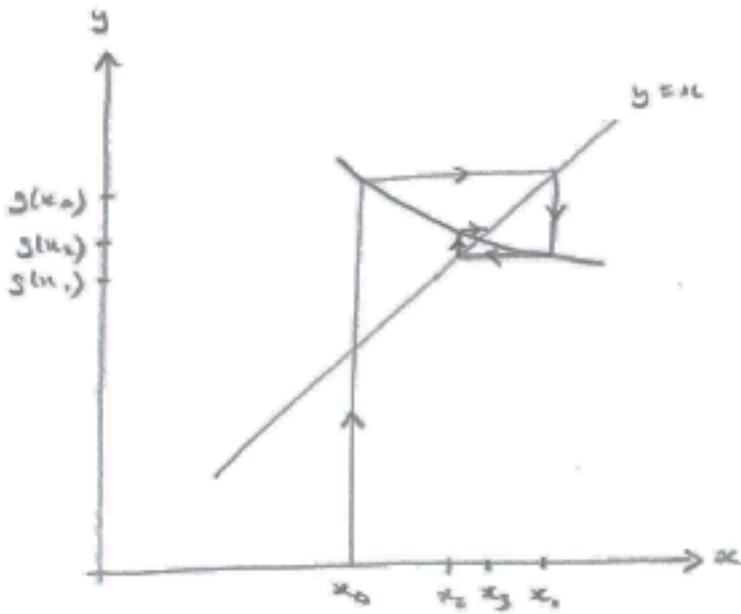
where each error term is proportional to the previous error term.

So, if the error is supposed to be getting smaller, we want $|g'(\alpha)| < 1$ (and the smaller the better).

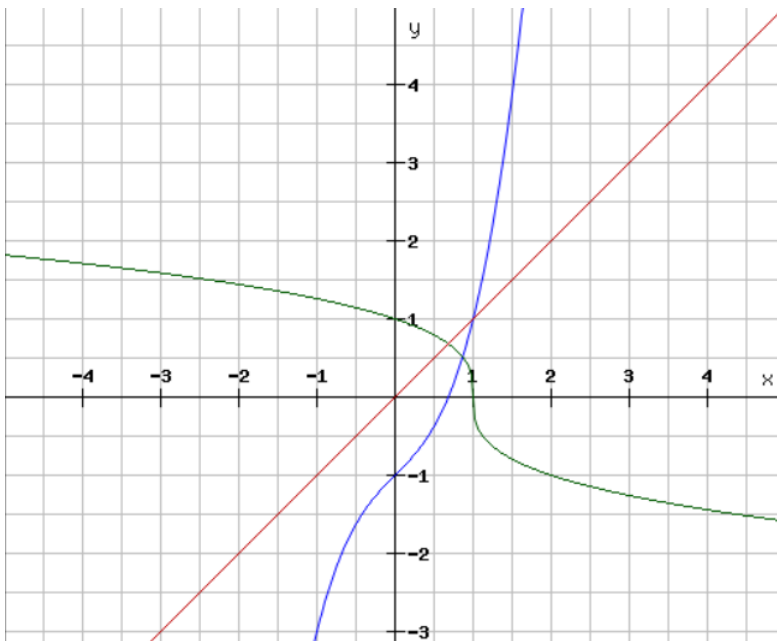
Referring to the diagram in (5), we can see whether $g'(\alpha) < 1$, by comparing the slopes of $y = g(x)$ and $y = x$ at $x = \alpha$. By imagining the line perpendicular to $y = x$ at $x = \alpha$, we can establish whether $g'(\alpha) > -1$.

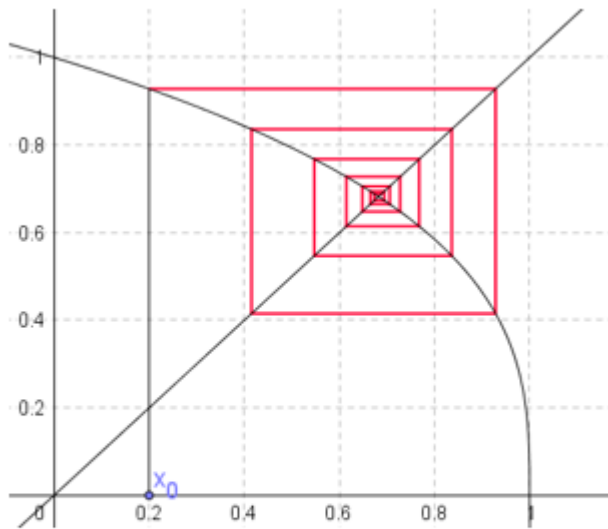
In the case of (B) above, $0 < g'(\alpha) < 1$, so that each error e_r is smaller than the previous one, and of the same sign. This gives rise to the staircase pattern.

In cases where $-1 < g'(\alpha) < 0$, the signs of the errors alternate, and a 'cobweb' pattern is obtained, as shown below.



An example of this occurs for the solution of $x^3 + x - 1 = 0$, when $g(x) = \sqrt[3]{1-x}$, as shown below:





$$g(x) = \sqrt[3]{1-x}$$

(7) The above condition for convergence can be applied without having to draw any graphs, if we have a rough estimate of the solution.

Thus, for the rearrangements (A), (B) and (C), we can obtain $g'(1)$, for example. The accurate figure of $g'(1.32472)$ is also shown, for comparison.

$$(A): g'(x) = 3x^2 \Rightarrow g'(1) = 3 \text{ \& } g'(1.32472) = 5.26$$

$$(B): g'(x) = \frac{1}{3} (x+1)^{-\frac{2}{3}} \Rightarrow g'(1) = 0.21 \text{ \& } g'(1.32472) = 0.19$$

$$(C): g'(x) = -x^{-2} - 2x^{-3} \Rightarrow g'(1) = -3 \text{ \& } g'(1.32472) = -1.43$$

Thus, in the case of (B), the fact that $-1 < 0.21 < 1$ strongly suggests convergence, whilst convergence is very unlikely in the cases of (A) and (C).

(8) The fixed point method provides an automatic interval for the solution where $-1 < g'(\alpha) < 0$; namely (x_{r-1}, x_r) or (x_r, x_{r-1}) , depending on whether $x_{r-1} < x_r$.

In order to obtain an interval for cases where $0 < g'(\alpha) < 1$, we take x_r as one of the bounds (lower or upper, depending on whether $x_r < \alpha$) and then estimate a suitable value for the other bound, checking it by the Change of Sign method.

Thus for (B) above, we could take 1.324 as the lower bound, and 1.325, as $f(1.324) = 1.324^3 - 1.324 - 1 = -0.0031 < 0$ and $f(1.325) = 1.325^3 - 1.325 - 1 = 0.0012 > 0$

(9) Number of iterations needed

For (B), with $x_0 = 1$, it is clear after a few iterations that

$$\alpha \approx 1.32, \text{ so that } e_0 \approx 1 - 1.32 = -0.32$$

$$\text{Also } g'(1.32) = 0.19 \text{ (2dp)} \text{ and } e_{r+1} \approx g'(\alpha) \cdot e_r,$$

$$\text{so that } e_r \approx -0.32(0.19)^r$$

$$\text{Thus, if } r = 8, e_r \approx -5 \times 10^{-7}$$

The actual value of e_r is $1.324715 - 1.324717957$

$$= -3 \times 10^{-6} \text{ (6dp)}, \text{ so the approximation is not especially good.}$$

If we want to find the approximate number of iterations needed in order to achieve a particular level of error, then we can take logs.

Thus, to achieve an error of approximately -5×10^{-6} (giving an answer approximately correct to 5dp):

$$-5 \times 10^{-6} = -0.32(0.19)^r$$

$$\Rightarrow 1.5625 \times 10^{-5} = (0.19)^r$$

$$\Rightarrow \log_{10}(1.5625) - 5 = r \log_{10}(0.19)$$

$$\Rightarrow r = 6.7$$

ie 7 iterations will be required.

(10) Quick way of estimating α

$$\text{As } e_{r+1} \approx g'(\alpha) \cdot e_r,$$

$$x_{r+1} - \alpha \approx g'(\alpha)(x_r - \alpha)$$

Then, given x_r & x_{r+1} , α can be found approximately if an estimate of $g'(\alpha)$ is available.

To find $g'(\alpha)$:

(a) base on an estimate of α (as in (9))

or (b) find an approximation to the gradient at α :

$$g'(\alpha) \approx \frac{g(x_{r+1}) - g(x_r)}{x_{r+1} - x_r} = \frac{x_{r+2} - x_{r+1}}{x_{r+1} - x_r}$$

This is referred to as the 'ratio of differences', and a more accurate value for $g'(\alpha)$ can be obtained by increasing r .

x_r for (B)	$x_{r+1} - x_r$	$\frac{x_{r+2} - x_{r+1}}{x_{r+1} - x_r}$
1	0.259921	
1.25992	0.052373	0.201495
1.31229	0.010060	0.192084
1.32235	0.001915	0.190351
1.32427	0.000364	0.190023
1.32463	0.000069	0.189961
1.32470	0.000013	0.189950
1.32472	0.000002	
1.32472		

The ratio of differences can of course be presented in the form

$\frac{x_{r+1}-x_r}{x_r-x_{r-1}}$ instead (where the value of r has been increased by 1).

(11) Alternative derivation of the ratio of differences result:

$$x_r = \alpha + e_r \quad \text{and} \quad e_r \approx k e_{r-1}$$

$$\frac{x_{r+1}-x_r}{x_r-x_{r-1}} = \frac{(\alpha+e_{r+1})-(\alpha+e_r)}{(\alpha+e_r)-(\alpha+e_{r-1})}$$

$$= \frac{e_{r+1}-e_r}{e_r-e_{r-1}} = \frac{k e_r - e_r}{k e_{r-1} - e_{r-1}}$$

$$= \frac{(k-1)e_r}{(k-1)e_{r-1}} \approx k$$

$$\text{ie } \frac{x_{r+1}-x_r}{x_r-x_{r-1}} \approx k$$

(12) What if $g'(\alpha) = 0$?

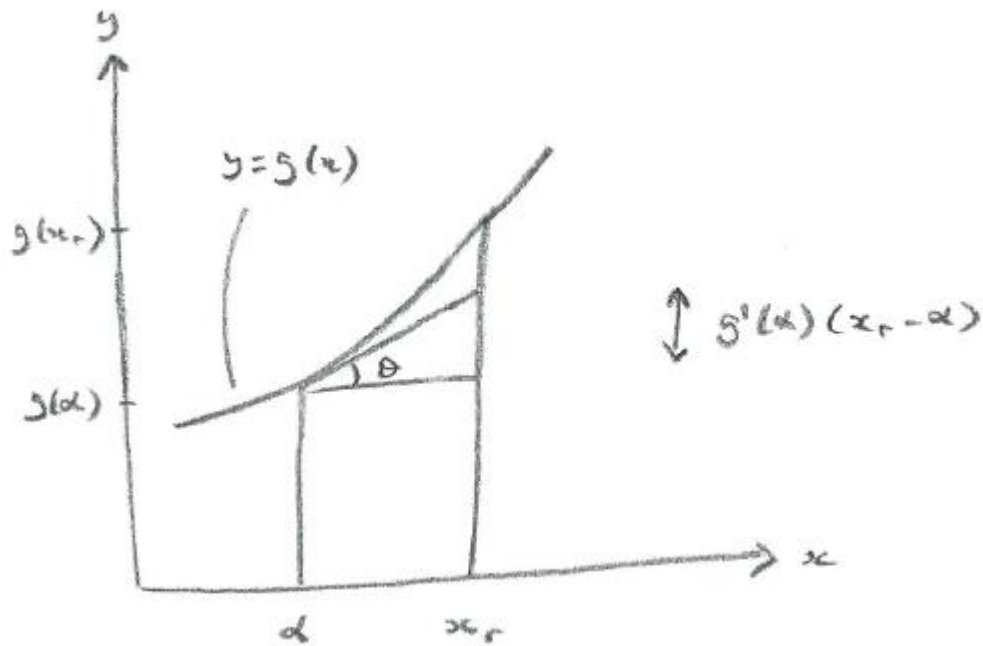
$e_{r+1} \approx g'(\alpha) \cdot e_r$ becomes $e_{r+1} \approx 0$, which isn't very helpful.

$g'(\alpha) \approx \frac{g(\alpha)-g(x_r)}{\alpha-x_r}$ can be written as

$$g(x_r) \approx g(\alpha) + g'(\alpha)(x_r - \alpha) \quad (\text{A})$$

Referring to the diagram below, an approximate value for $g(x_r)$ is obtained by adding $\tan\theta(x_r - \alpha)$ to $g(\alpha)$, where $\tan\theta = g'(\alpha)$

[in this case, it gives an underestimate]



(A) gives the first two terms of the Taylor expansion of $g(x_r)$ about α

The 3rd term is $g''(\alpha) \frac{(x_r - \alpha)^2}{2!}$

Then if $g'(\alpha) = 0$,

$$g(x_r) \approx g(\alpha) + g''(\alpha) \frac{(x_r - \alpha)^2}{2!}$$

so that $e_{r+1} = x_{r+1} - \alpha = g(x_r) - g(\alpha) \approx \frac{1}{2} g''(\alpha) e_r^2$ (B)

Thus each error term is proportional to the square of the previous error term. This is referred to as **quadratic or 2nd order convergence**.

From (B), if $g''(\alpha) > 0$, the errors after e_0 will all be positive; ie giving a staircase pattern. If $g''(\alpha) < 0$, the errors after e_0 will all be negative; again giving a staircase pattern.

(13) Relaxed iteration

Consider $h(x) = (1 - \lambda)x + \lambda g(x)$

$$h(\alpha) = (1 - \lambda)\alpha + \lambda g(\alpha) = (1 - \lambda)\alpha + \lambda\alpha = \alpha$$

Thus, $h(x)$ fulfills the same role as $g(x)$ and, with a suitable λ , it may be possible to convert an unfavourable $g(x)$, where there isn't convergence, to a favourable $h(x)$, where there is convergence. Where there is already convergence for $g(x)$, it may be possible to obtain faster convergence with $h(x)$.

A spreadsheet can be used to do this, as shown below (with k instead of λ).

B6 : ✕ ✓ *f_x* =(1-B\$2)*A6+B\$2*((A6^3)-1)

	A	B	C	D	E	F	G	H
1	(A) $g(x) = x^3 - 1$			(A) $g(x) = x^3 - 1$			(A) $g(x) = x^3 - 1$	
2	k	0.2		k	-0.1		k	-0.3
3								
4	x	h(x)		x	h(x)		x	h(x)
5	1	0.8		1	1.1		1	1.3
6	0.8	0.5424		1.1	1.1769		1.3	1.3309
7	0.5424	0.265835		1.1769	1.231578		1.3309	1.322945
8	0.265835	0.016425		1.231578	1.267932		1.322945	1.32521
9	0.016425	-0.18686		1.267932	1.290886		1.32521	1.32458
10	-0.18686	-0.35079		1.290886	1.304863		1.32458	1.324756
11	-0.35079	-0.48927		1.304863	1.313175		1.324756	1.324707
12	-0.48927	-0.61484		1.313175	1.318045		1.324707	1.324721
13	-0.61484	-0.73836		1.318045	1.320873		1.324721	1.324717
14	-0.73836	-0.87119		1.320873	1.322507		1.324717	1.324718
15	-0.87119	-1.02919		1.322507	1.323448		1.324718	1.324718
16	-1.02919	-1.24139		1.323448	1.323989		1.324718	1.324718
17	-1.24139	-1.57572		1.323989	1.3243		1.324718	1.324718
18	-1.57572	-2.24304		1.3243	1.324478		1.324718	1.324718
19	-2.24304	-4.25147		1.324478	1.32458		1.324718	1.324718
20	-4.25147	-18.9702		1.32458	1.324639		1.324718	1.324718

Alternatively, suppose that we are able to estimate α , and hence calculate $g'(\alpha)$ approximately.

$$\text{Then } h'(\alpha) = 0 \Rightarrow (1 - \lambda) + \lambda g'(\alpha) = 0$$

$$\Rightarrow \lambda(g'(\alpha) - 1) = -1$$

$$\Rightarrow \lambda = \frac{1}{1-g'(\alpha)}$$

For example, we can consider the three earlier rearrangements of $x^3 - x = 1$, and assume that it is known that a root lies between 1 and 2, with $x_0 = 1$.

$$(A) \ x = x^3 - 1 ; g'(1) = 3; \lambda = \frac{1}{1-g'(1)} = -0.5$$

$$(B) \ x = \sqrt[3]{x+1} ; g'(1) = 0.21; \lambda = \frac{1}{1-g'(1)} = 1.26582$$

$$(C) \ x = \frac{1}{x} + \frac{1}{x^2} ; g'(1) = -3; \lambda = \frac{1}{1-g'(1)} = 0.25$$

Convergence, or otherwise, of $h(x)$ can now be investigated for the calculated value of λ , using a spreadsheet:

(A)		(B)		(C)	
x	h(x)	x	h(x)	x	h(x)
1	1.5	1	1.329014	1	1.25
1.5	1.0625	1.329014	1.324608	1.25	1.2975
1.0625	1.494019	1.324608	1.324721	1.2975	1.314303
1.494019	1.073635	1.324721	1.324718	1.314303	1.320669
1.073635	1.491667	1.324718	1.324718	1.320669	1.323135
1.491667	1.077968			1.323135	1.324097
1.077968	1.490645			1.324097	1.324475
1.490645	1.079845			1.324475	1.324622
1.079845	1.490183			1.324622	1.32468
1.490183	1.080691			1.32468	1.324703
1.080691	1.489971			1.324703	1.324712
1.489971	1.081079			1.324712	1.324716
1.081079	1.489873			1.324716	1.324717
1.489873	1.081258			1.324717	1.324718
1.081258	1.489827			1.324718	1.324718

As can be seen, the calculated values of λ are effective for (B) and (C), but not for (A). In the case of (B), there was already convergence with $g(x)$, but $h(x)$ gives a faster convergence. In the case of (C), there previously wasn't any convergence.

For (A), the situation can be improved by taking the average of $g'(1) = 3$ and $g'(2) = 12$ (as the root lies between 1 and 2).

Then we can take $\lambda = \frac{1}{1 - \frac{1}{2}(3+12)} = -0.28571$

This gives the following convergence:

x	h(x)
1	1.285714
1.285714	1.331529
1.331529	1.323177
1.323177	1.325052
1.325052	1.324645
1.324645	1.324734
1.324734	1.324714
1.324714	1.324719
1.324719	1.324718
1.324718	1.324718
1.324718	1.324718