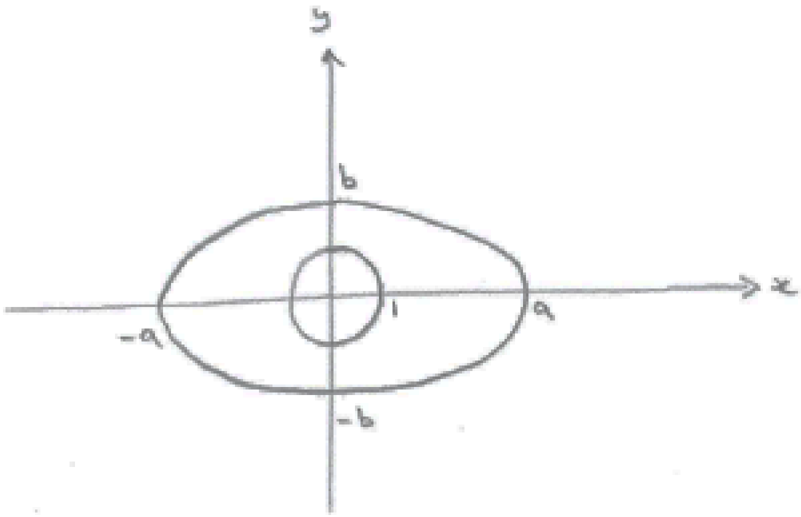
# Ellipses (7 pages; 18/8/19)

See "Conics" for features that are common to parabolas, ellipses and hyperbolas (as well as circles).

(1) If the circle $x^2 + y^2 = 1$ is stretched by a scale factor $a$ in the $x$ direction, and also by a scale factor $b$ in the $y$ direction, then the transformed curve is obtained by replacing $x$ by $\frac{x}{a}$ and replacing $y$ by $\frac{y}{b}$, to give $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the equation of an ellipse, centred on the Origin, passing through the points $(a, 0)$ and $(0, b)$.




If (as here) $a > b$, then $a$ is termed the **semi-major axis** and $b$ is the **semi-minor axis** (even though $a$ & $b$ are lengths rather than axes!)

(2) Parametric equations

For a circle of radius $a$, centre the Origin, $x^2 + y^2 = a^2$

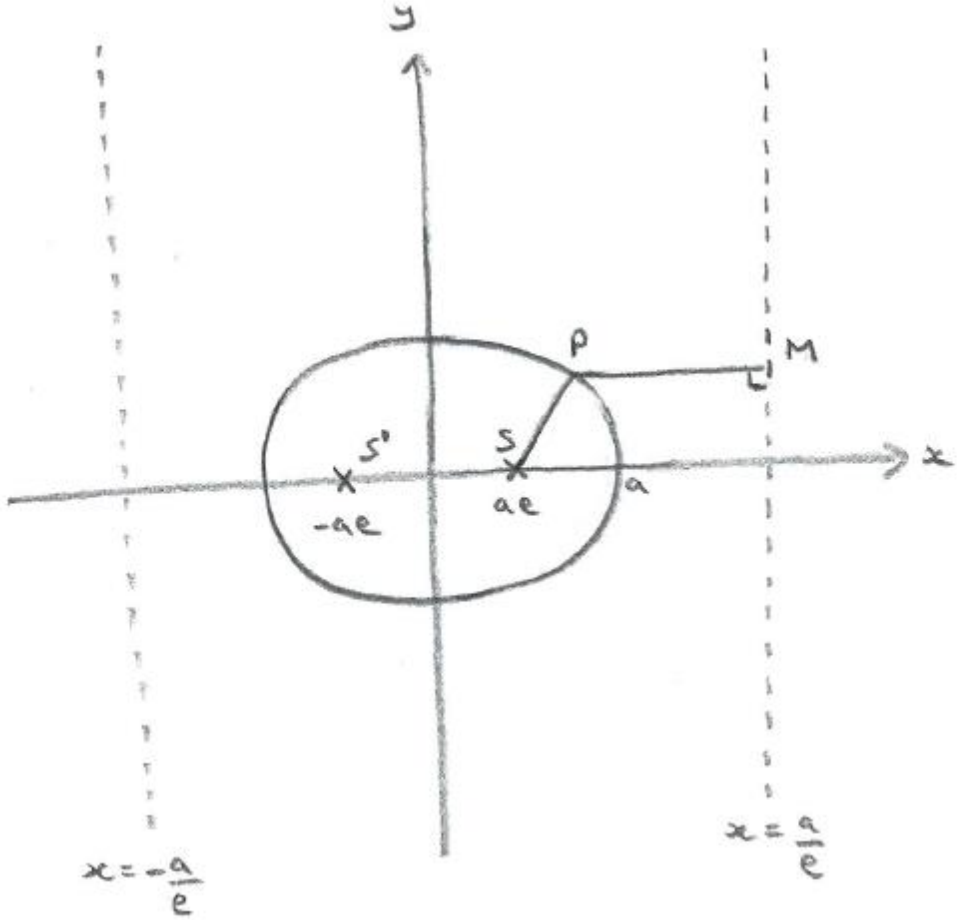
or $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$

The parametric equations of this circle are $x = acos\theta, y = asin\theta$.

The parametric equations of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are

$x = acos\theta, y = bsin\theta$.

(3) For the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, suppose that the **eccentricity** $e$ is defined by $b^2 = a^2(1-e^2)$, and that one focus is taken to be the point $(0, ae)$, whilst one directrix is taken to be the line $x = \frac{a}{e}$



**Exercise**: Show that $\mathrm{e} = \frac{PS}{PM}$ , where P is the general point $(x, y)$, with S and M being defined as in "Conics", section (2).

**Solution**

$$\left(\frac{PS}{PM}\right)^2 = \frac{(x-ae)^2+y^2}{\left(x-\frac{a}{e}\right)^2}$$

Using the parametric form $x = acos\theta$ , $y = bsin\theta$ ,

$$\frac{(x-ae)^2+y^2}{\left(x-\frac{a}{e}\right)^2} = \frac{a^2(cos\theta-e)^2+a^2(1-e^2)sin^2\theta}{a^2\left(cos\theta-\frac{1}{e}\right)^2}$$

$$= \frac{e^2[(cos\theta-e)^2+\left(1-e^2\right)sin^2\theta]}{(ecos\theta-1)^2} = \frac{e^2[cos^2\theta-2ecos\theta+e^2+sin^2\theta-e^2sin^2\theta]}{(ecos\theta-1)^2}$$

$$= \frac{e^2[1-2ecos\theta+e^2cos^2t]}{(ecos\theta-1)^2} = e^2$$

Also, from $b^2 = a^2(1-e^2)$, we can see that $e < 1$.

It can be shown similarly that the focus could also be taken to be the point $(0, -ae)$, with the directrix being the line $x = -\frac{a}{e}$

**Notes**

(i) In order to recall the definition $\mathrm{e} = \frac{PS}{PM}$ (ie to establish which way round PS and PM are), consider the point $(-a, 0)$ for the ellipse shown below, using the focus at $(ae, 0)$ and the directrix $x = \frac{a}{e}$. Clearly, $PS < PM$, and we know that $e < 1$ for an ellipse, so it must be the case that $\mathrm{e} = \frac{PS}{PM}$.

(ii) When P is $(a, 0)$, $\frac{PS}{PM} = \frac{a-ae}{\frac{a}{e}-a} = \frac{1-e}{\frac{1}{e}-1} = \frac{e(1-e)}{1-e} = e$

Also, when P is $(0, b)$, $\frac{PS}{PM} = \frac{\sqrt{(ae)^2+b^2}}{\frac{a}{e}} = \frac{\sqrt{(ae)^2+a^2(1-e^2)}}{\frac{a}{e}} = e$

(iii) When $e = \frac{1}{2}$ (for example), $b^2 = a^2(1 - e^2) = \frac{3a^2}{4}$, so that $b = \frac{\sqrt{3}}{2}a$

(4) Circle (e = 0)

Referring to the diagram for the ellipse, as $e \to 0$, the foci tend to the Origin and the directrices tend to the lines $x = \infty$ and

$x = -\infty$.

(5) Equations of tangents and normals to an ellipse at the point with parameter $\theta$

$x = acos\theta, y = bsin\theta$ , so that $\frac{dx}{d\theta} = -asin\theta$, $\frac{dy}{d\theta} = bcos\theta$

Hence the gradient of the tangent is $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{bcos\theta}{-asin\theta}$

and the equation of the tangent is $\frac{y-bsin\theta}{x-acos\theta} = -\frac{bcos\theta}{asin\theta}$

or $yasin\theta - absin^2\theta = -xbcos\theta + abcos^2\theta$,

and $xbcos\theta + yasin\theta = ab$ (as $sin^2\theta + cos^2\theta = 1$),

or $\frac{xcos\theta}{a} + \frac{ysin\theta}{b} = 1$

[compare with the equivalent form for the hyperbola:

$\frac{xsec\theta}{a} - \frac{ytan\theta}{b} = 1$; see "Hyperbolas"]

**Exercise**: Find the equation of the normal at the same point

**Solution**

The gradient of the normal is $\frac{asin\theta}{bcos\theta}$, so that the equation of the normal is $\frac{y-bsin\theta}{x-acos\theta} = \frac{asin\theta}{bcos\theta}$

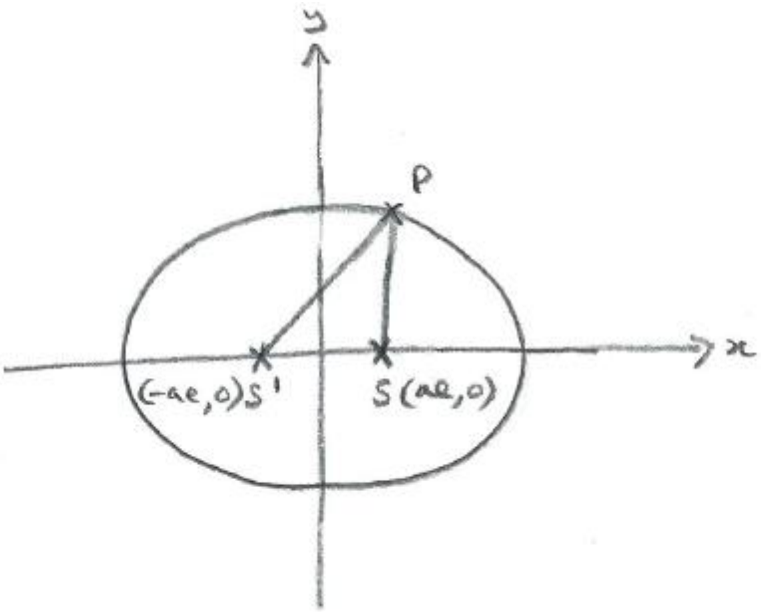
or $ybcos\theta - b^2 sin\theta cos\theta = xasin\theta - a^2 sin\theta cos\theta$

and $\frac{yb}{sin\theta} - \frac{xa}{cos\theta} = b^2 - a^2$

(6) If the tangent is to have a specified gradient ($m$), then we can find the (two) possible points on the ellipse by proceeding as if to find the intersection of the straight line, $y = mx + c$, with the ellipse, and choosing $c$ such that the discriminant of the resulting quadratic is zero (so that the line meets the ellipse as a tangent).

(7) Alternative definition of ellipse

If P is a general point on an ellipse with foci at $S: (ae, 0)$ and $S': (-ae, 0)$, then we can show that $PS + PS'$ is a constant.




$PS + PS' = \sqrt{(x-ae)^2 + y^2} + \sqrt{(x+ae)^2 + y^2}$

Using the parametric form $x = acos\theta, y = bsin\theta$, and the fact that $b^2 = a^2(1-e^2)$,

$PS + PS' =$

$\sqrt{(acos\theta - ae)^2 + b^2sin^2\theta} + \sqrt{(acos\theta + ae)^2 + b^2sin^2\theta}$

$= a\sqrt{(cos\theta - e)^2 + (1-e^2)sin^2\theta}$

$+a\sqrt{(cos\theta + e)^2 + (1-e^2)sin^2\theta}$

$= a\sqrt{cos^2\theta - 2ecos\theta + e^2 + sin^2\theta - e^2sin^2\theta}$

$+a\sqrt{cos^2\theta + 2ecos\theta + e^2 + sin^2\theta - e^2sin^2\theta}$

$= a\sqrt{1 - 2ecos\theta + e^2cos^2\theta} + a\sqrt{1 + 2ecos\theta + e^2cos^2\theta}$

$= a(1 - ecos\theta) + a(1 + ecos\theta) = 2a$

[Check: Consider P to be $(a, 0)$, so that

$PS + PS' = (a + ae) + (a - ae) = 2a$]

(8) Polar equation of ellipse

See "Conics" for the derivation of the polar form of a general conic: $r = \frac{ep}{1+ecos\theta}$ , where $p$ is the (positive) distance between the focus and the directrix, for the case where the directrix is vertical and lies to the right of the pole.

In order for the ellipse to have the same location and orientation as the general conic used to derive the above formula, we make a translation of $ae$ to the left (so that the right-hand focus is moved to the Origin).

For this ellipse, $p = \frac{a}{e} - ae$ , so that

$r = \frac{ep}{1+ecos\theta}$ becomes $r = \frac{a(1-e^2)}{1+ecos\theta}$

Note that we could have chosen instead to position the left-hand focus at the origin; in which case the equation becomes

$r = \frac{a(1-e^2)}{1-ecos\theta}$