Differentiation (4 pages; 3/9/18)

(1) Derivative of e^x

(i) Consider the example of compound interest, where interest is added at the rate of $100p\%$ pa, so that an amount x grows to $x(1+p)^t$ after t years.

If t is replaced by the small period δt,

the increase in x over the period δt is $x(1+p)^{\delta t} - x$

$= x\left(1 + p\delta t + o(\delta t)\right) - x = xp\delta t + o(\delta t)$ (where $|p| < 1$)

[where $o(\delta t)$ means "terms of order smaller than δt" (ie involving $(\delta t)^2$ and higher powers)]

Thus $\frac{dx}{dt} = \lim_{\delta t \to 0} \frac{xp\delta t + o(\delta t)}{\delta t} = px$

This has also been found to be an appropriate model for population growth; ie $\frac{dP}{dt} = kP$, and Newton's law of cooling is $\frac{dT}{dt} = -kT$

(ii) The function $y = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ can easily be seen to have the property that $\frac{d}{dx}(e^x) = e^x$, on differentiating term by term. And, by the Chain rule, $\frac{d}{dx}(e^{kx}) = ke^{kx}$.

Thus a solution of $\frac{dP}{dt} = kP$, for example, is $P = P_0 e^{kt}$, with P_0 being $P(0)$, the initial population.

(iii) The function $y = e^x$ is the special case of the family $y = a^x$ where the gradient at $x = 0$ is 1 (for the purpose of sketching $y = e^x$).

It is shown later on that $\frac{d}{dx}(a^x) = lna. a^x$

It is possible to express the solution of $\frac{dP}{dt} = kP$ as $P = P_0 a^{\lambda t}$, as follows:

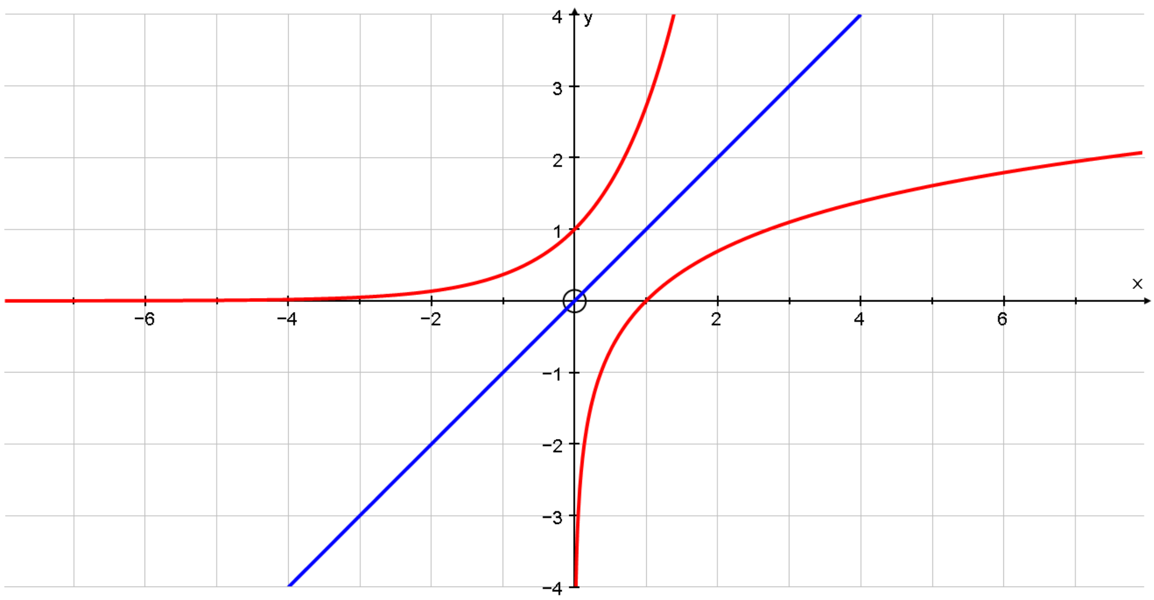
Let $P = P_0 a^{\lambda t}$, so that $\frac{dP}{dt} = \lambda lna. P_0 a^{\lambda t}$

Then let $k = \lambda lna$, to give $P = P_0 a^{(\frac{kt}{lna})}$

Thus e is just the value of a that gives the simplest from of the solution.

(2) Derivative of $\boldsymbol{ln\ x}$

$y = lnx$ is the inverse function of $y = e^x$, and therefore its reflection in the line $y = x$



The gradient of one of these two curves is the reciprocal of the other at the reflected point. (Consider, for example, what is happening when $x = 2$ for $y = lnx$: $y = 2$ at the reflected point on $y = e^x$, and the tangents to the two curves have reciprocal gradients, because the roles of x and y are being reversed.)

Suppose that $e^a = b$, so that $a = lnb$. (Consider, for example, where $b = 2$ and $a = 0.693$ (3sf).)

Then $\frac{d}{dx}(lnx)|_{x=b} = \frac{1}{\frac{d}{dx}(e^x|_{x=a})} = \frac{1}{e^a} = \frac{1}{b}$

Since this is true for all values of x in the domain of $y = lnx$, we can say that $\frac{d}{dx}(lnx) = \frac{1}{x}$

Alternative approach 1 (informal)

When $y = e^x$, $\frac{\mathrm{d}y}{\mathrm{d}x} = y$. In order to obtain the derivative of lnx, we reverse the roles of x & y, so that $\frac{dx}{dy} = x$, giving $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$

Alternative approach 2

$y = e^x \Rightarrow x = lny$

Then, differentiating both sides wrt x,

$1 = \frac{d}{dy}lny\,.\frac{dy}{dx} \Rightarrow \frac{d}{dy}lny = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{e^x} = \frac{1}{y}$

Relabelling we then have $\frac{d}{dx}lnx = \frac{1}{x}$

(3) Differentiating a^x (two methods)

(a) Let $y = a^x$

Then $lny = xlna$

Differentiating implicitly gives $\frac{1}{y}\frac{dy}{dx} = lna$,

so that $\frac{dy}{dx} = lna.a^x$ (*)

(b) Let $y = a^x$ and let $a = e^b$

(it is assumed that $a > 0$; the result (*) also implies this)

Then $y = (e^b)^x = e^{bx}$

and $\frac{dy}{dx} = be^{bx} = lna.a^x$

Note: When $a = e, lna = 1$ (which may help to recall the result).