

Differentiation - Exercises (Sol'ns)(6 pages; 19/10/18)

(1) Find the derivative of $\tan x$ using (a) the Quotient rule, and
(b) the Product rule

Solution

$$\begin{aligned} \text{(a)} \quad \frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} = (\cos^2 x + \sin^2 x)\sec^2 x \\ &= \sec^2 x \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{d}{dx}(\tan x) &= \frac{d}{dx}(\sin x \cdot (\cos x)^{-1}) \\ &= \cos x(\cos x)^{-1} + (\sin x)(-1)(\cos x)^{-2}(-\sin x) \\ &= 1 + \tan^2 x = \sec^2 x \end{aligned}$$

(2) Find the turning points of $y = (x^2 - 4x + 3)^2$

Solution

Method 1

$$\text{As } x^2 - 4x + 3 = (x - 1)(x - 3),$$

$$y = (x - 1)^2(x - 3)^2$$

$$\text{Then } \frac{dy}{dx} = 2(x - 1)(x - 3)^2 + (x - 1)^2(2)(x - 3)$$

$$= 2(x - 1)(x - 3)(x - 3 + x - 1)$$

$$= 4(x - 1)(x - 3)(x - 2)$$

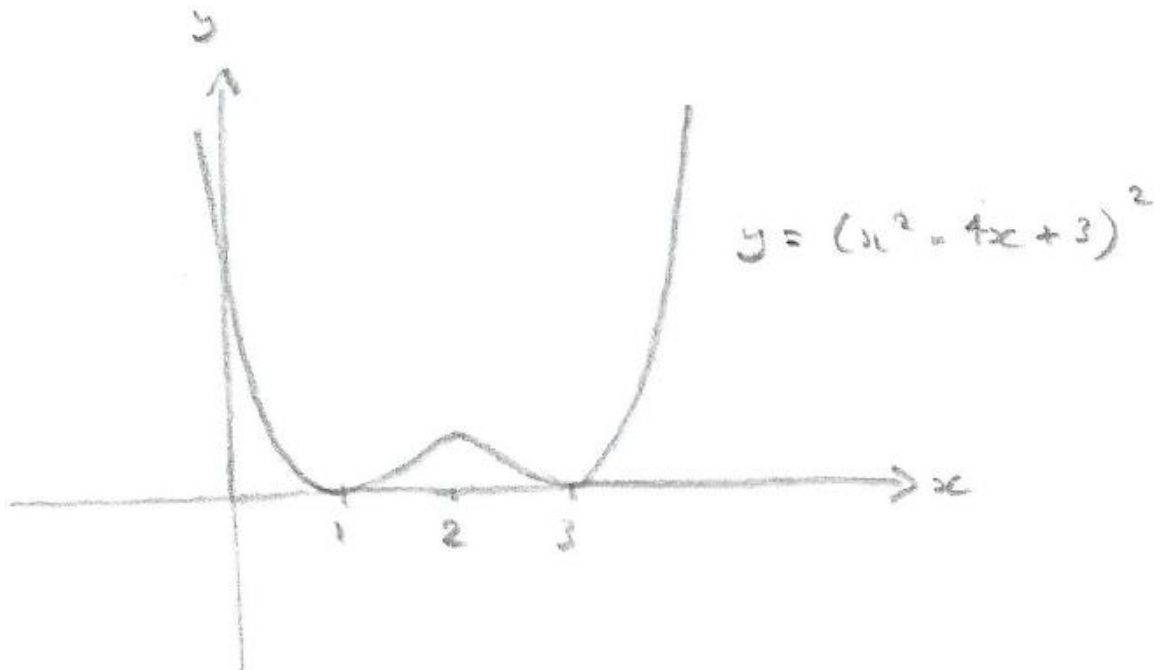
$$\frac{dy}{dx} = 0 \text{ when } x = 1, 2 \text{ \& } 3$$

At $x = 1$, $\frac{dy}{dx}$ changes from -ve to +ve, indicating a min. point.

At $x = 2$, $\frac{dy}{dx}$ changes from +ve to -ve, indicating a max. point.

At $x = 3$, $\frac{dy}{dx}$ changes from -ve to +ve, indicating a min. point.

The min. points are therefore $(1, 0)$ and $(3, 0)$, whilst the max. is at $(2, 1)$.



Method 2

$(x^2 - 4x + 3)^2 \geq 0$ and $(x^2 - 4x + 3)^2 = (x - 1)^2(x - 3)^2 = 0$ has roots at $x = 1$ & 3 , so that there are minima at these two points.

For $x = 1 - t$, $y = x^2 - 4x + 3$ and hence $y = (x^2 - 4x + 3)^2$ increases as t increases, and similarly for $x = 3 + t$.

For $1 < x < 3$, $y = (x^2 - 4x + 3)^2$ attains a max. when

$x^2 - 4x + 3$ (which is negative in this range) is at a min. ; ie when $x = 2$.

(3) If $f(x) = x^2$, what is $f'(3x)$?

Solution

Method 1

Note that the differentiation is wrt $3x$ (rather than x).

Let $u = 3x$. Then $f'(3x) = f'(u) = \frac{d}{du}(u^2) = 2u = 2(3x) = 6x$

Method 2

$f'(x) = 2x \Rightarrow f'(3x) = 2(3x) = 6x$

(4) Find $\frac{d}{dx}(x^{\sin x})$

Solution

$$\begin{aligned} \frac{d}{dx}(x^{\sin x}) &= \frac{d}{dx}(e^{\ln x \cdot \sin x}) = e^{\ln x \cdot \sin x} \left(\frac{1}{x} \sin x + \ln x \cdot \cos x \right) \\ &= x^{\sin x} \left(\frac{1}{x} \sin x + \ln x \cdot \cos x \right) \end{aligned}$$

(5) Find $\frac{d}{dx}(a^x)$

Solution

Method 1

Let $a = e^b$. Then $\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{bx}) = be^{bx} = \ln a \cdot a^x$

Method 2

Let $y = a^x$. Then $\ln y = x \ln a$,

and, differentiating wrt x gives $\frac{1}{y} \frac{dy}{dx} = \ln a$, so that $\frac{dy}{dx} = \ln a \cdot a^x$

(6) (i) Give a definition for a point of inflexion.

(ii) Give an example of a turning point where $f''(x) = 0$

Solution

(i) A turning point of the gradient.

(ii) $f(x) = x^4$

(7) For a particular point on a curve:

S = stationary point

T = turning point

PI = point of inflexion

2D0 = 2nd derivative is zero

TG = turning point of gradient

Which of the following are true?

(a) $T \Rightarrow S$

(b) $S \Rightarrow T$

(c) $PI \Rightarrow S$

(d) $2D0 \Leftrightarrow PI$

(e) $PI \Leftrightarrow TG$

(f) $TG \Rightarrow 2D0$

Solution

(a) $T \Rightarrow S$ (true)

(b) $S \Rightarrow T$ (false; eg $y = x^3$ at $x = 0$)

(c) $PI \Rightarrow S$ (false: eg $y = \tan x$ at $x = 0$)

(d) $2D0 \Leftrightarrow PI$ (false; eg $y = x^4$ at $x = 0$)

(e) $PI \Leftrightarrow TG$ (true)

(f) $TG \Rightarrow 2D0$ (true)

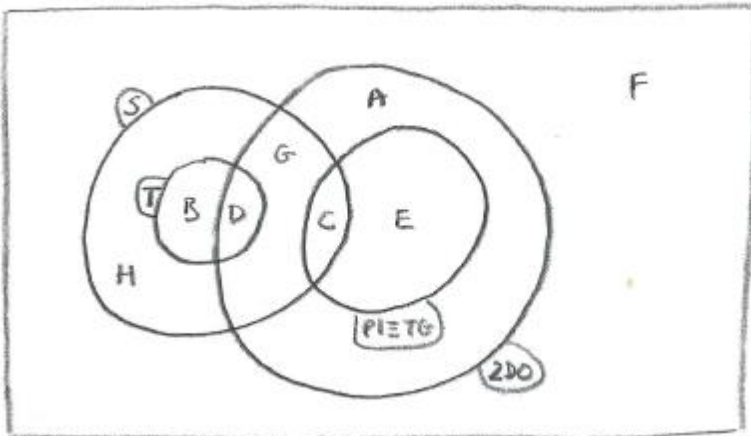
(8) Referring to the abbreviations in (7), represent the events S, T, PI, 2D0 & TG of (ii) in a Venn diagram, showing where the following functions lie:

$A: y = x$ $B: y = x^2$ $C: y = x^3$ $D: y = x^4$ $E: y = \tan x$ $F: y = e^x$

(ie depending on whether these functions exhibit any of the events)

Are there any regions of the Venn diagram that aren't satisfied by any functions?

Solution



Region G would be satisfied by, for example, the function

$y = x^3$ with domain limited to $x \geq 0$, but apart from this type of example a stationary point means either a turning point or a point of inflexion, so that region H is empty.

(9) If $f(x) = \sin x$, express $f^{(n)}(0)$ in terms of n , when n is odd (where $f^{(n)}(x)$ denotes the n th derivative of $f(x)$)

Solution

$$(-1)^{\frac{n-1}{2}}$$

(10) Show that $\int \frac{1}{\sqrt{1+a^2x^2}} dx = \frac{1}{a} \ln \left| \sqrt{1+a^2x^2} + ax \right| + c$, by differentiation

Solution

$$\frac{d}{dx} \left(\frac{1}{a} \ln \left| \sqrt{1+a^2x^2} + ax \right| \right) = \frac{1}{a} \cdot \frac{\frac{1}{2}(1+a^2x^2)^{-\frac{1}{2}}(2a^2x) + a}{\sqrt{1+a^2x^2} + ax}$$

Writing $A = \sqrt{1+a^2x^2}$, this gives:

$$\frac{A^{-1}ax+1}{A+ax} = \frac{1}{A} \cdot \frac{ax+A}{A+ax} = \frac{1}{A}, \text{ as required}$$