Differentiation - Exercises (Sol'ns) (6 pages; 18/2/20)

(1*) Find the derivative of $tanx$ using (a) the Quotient rule, and (b) the Product rule

Solution

(a) $\frac{d}{dx}(tanx) = \frac{d}{dx}\left(\frac{sinx}{cosx}\right) = \frac{cosx(cosx)-sinx(-sinx)}{cos^2x}$

$= (cos^2x + sin^2x)sec^2x$

$= sec^2x$

(b) $\frac{d}{dx}(tanx) = \frac{d}{dx}(sinx\,.\,(cosx)^{-1})$

$= cosx(cosx)^{-1} + (sinx)(-1)(cosx)^{-2}(-sinx)$

$= 1 + tan^2x = sec^2x$

(2***) Find the turning points of $y = (x^2 - 4x + 3)^2$

Solution

Method 1

As $x^2 - 4x + 3 = (x-1)(x-3)$,

$y = (x-1)^2(x-3)^2$

Then $\frac{dy}{dx} = 2(x-1)(x-3)^2 + (x-1)^2(2)(x-3)$

$= 2(x-1)(x-3)(x-3+x-1)$

$= 4(x-1)(x-3)(x-2)$

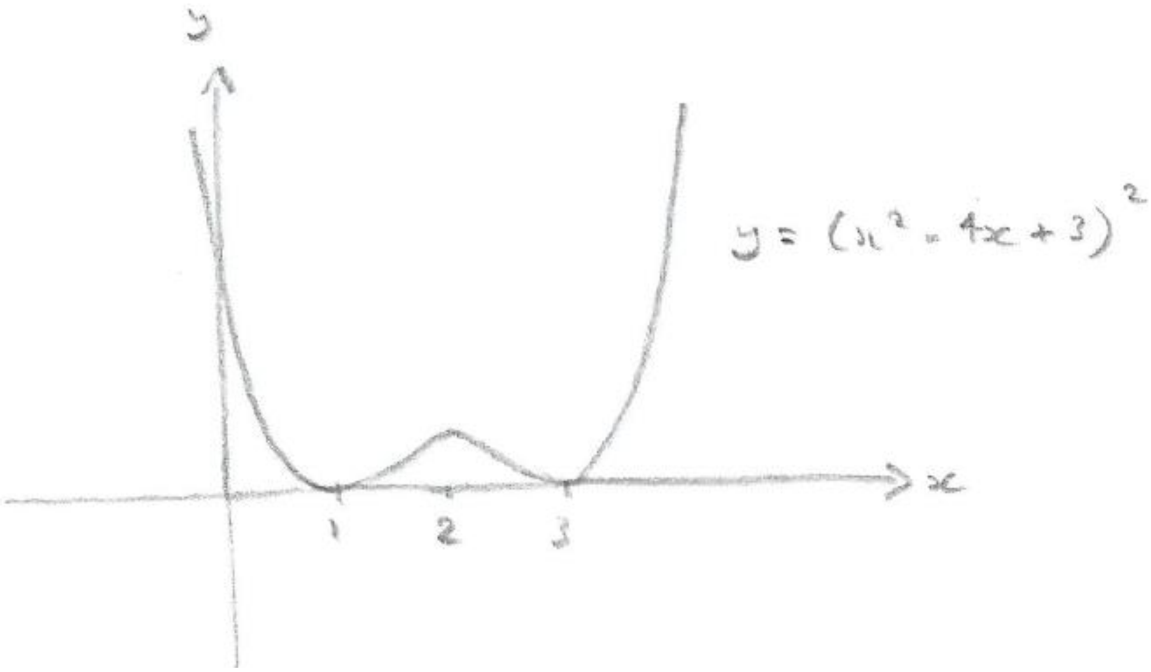
$\frac{dy}{dx} = 0$ when $x = 1, 2$ & 3

At $x = 1, \frac{dy}{dx}$ changes from –ve to +ve, indicating a min. point.

At $x = 2, \frac{dy}{dx}$ changes from +ve to -ve, indicating a max. point.

At $x = 3, \frac{dy}{dx}$ changes from –ve to +ve, indicating a min. point.

The min. points are therefore (1, 0) and (3, 0), whilst the max. is at (2,1).



Method 2

$(x^2 - 4x + 3)^2 \geq 0$ and $(x^2 - 4x + 3)^2 = (x - 1)^2(x - 3)^2 = 0$ has roots at $x = 1$ & 3, so that there are minima at these two points.

For $x = 1 - t$, $y = x^2 - 4x + 3$ and hence y = $(x^2 - 4x + 3)^2$ increases as t increases, and similarly for $x = 3 + t$.

For $1 < x < 3$, y = $(x^2 - 4x + 3)^2$ attains a max. when

$x^2 - 4x + 3$ (which is negative in this range) is at a min. ; ie when

$x = 2$.

(3**) If $f(x) = x^2$, what is $f'(3x)$?

Solution

Method 1

Note that the differentiation is wrt $3x$ (rather than x).

Let $u = 3x$. Then $f'(3x) = f'(u) = \frac{d}{du}(u^2) = 2u = 2(3x) = 6x$

Method 2

$f'(x) = 2x \Rightarrow f'(3x) = 2(3x) = 6x$

(4***) Find $\frac{d}{dx}\left(x^{sinx}\right)$

Solution

$\frac{d}{dx}\left(x^{sinx}\right) = \frac{d}{dx}\left(e^{lnx.sinx}\right) = e^{lnx.sinx}(\frac{1}{x}sinx + lnx.cosx)$

$= x^{sinx}(\frac{1}{x}sinx + lnx.cosx)$

(5***) Find $\frac{d}{dx}(a^x)$

Solution

Method 1

Let $a = e^b$. Then $\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{bx}) = be^{bx} = lna.a^x$

Method 2

Let $y = a^x$. Then $lny = xlna$,

and, differentiating wrt x gives $\frac{1}{y}\frac{dy}{dx} = lna$, so that $\frac{dy}{dx} = lna.a^x$

(6**) For a particular point on a curve:

S = stationary point

T = turning point

PI = point of inflexion

2D0 = 2nd derivative is zero

TG = turning point of gradient

Which of the following are true?

(a) $T \Rightarrow S$

(b) S $\Rightarrow T$

(c) $PI \Rightarrow S$

(d) $2D0 \Leftrightarrow PI$

(e) $PI \Leftrightarrow TG$

(f) $TG \Rightarrow 2D0$

Solution

(a) $T \Rightarrow S$ (true)

(b) S $\Rightarrow T$ (false; eg $y = x^3$ at $x = 0$)

(c) $PI \Rightarrow S$ (false: eg $y = tanx$ at $x = 0$)

(d) $2D0 \Leftrightarrow PI$ (false; eg $y = x^4$ at $x = 0$)

(e) $PI \Leftrightarrow TG$ (true)

(f) $TG \Rightarrow 2D0$ (true)

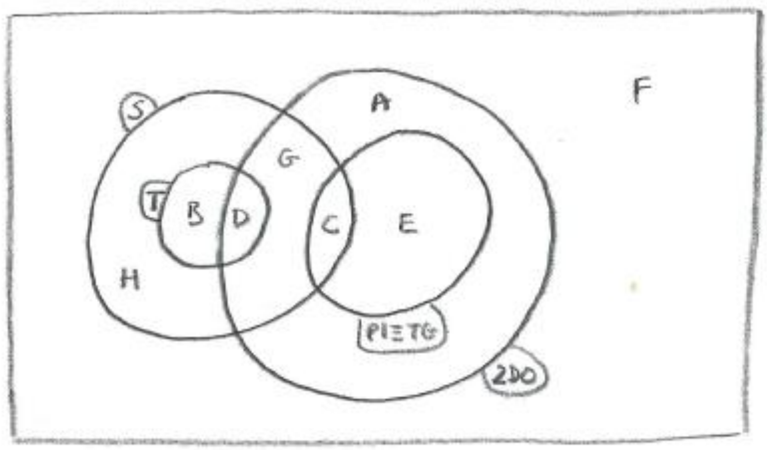
(7**) Referring to the abbreviations in (6), represent the events S, T, PI, 2D0 & TG of (ii) in a Venn diagram, showing where the following functions lie:

$A: y = x \quad B: y = x^2 \quad C: y = x^3 \quad D: y = x^4 \quad E: y = tanx \quad F: y = e^x$

(ie depending on whether these functions exhibit any of the events)

Are there any regions of the Venn diagram that aren't satisfied by any functions?

Solution



Region G would be satisfied by, for example, the function

$y = x^3$ with domain limited to $x \geq 0$, but apart from this type of example a stationary point means either a turning point or a point of inflexion, so that region H is empty.

(8**) If $f(x) = sinx$, express $f^{(n)}(0)$ in terms of n, when n is odd (where $f^{(n)}(x)$ denotes the nth derivative of $f(x)$)

Solution

$(-1)^{\frac{n-1}{2}}$

(9***) Show that $\int \frac{1}{\sqrt{1+a^2x^2}}dx = \frac{1}{a}ln\left|\sqrt{1+a^2x^2}\ +ax\right| + c$, by differentiation

Solution

$\frac{d}{dx}\left(\frac{1}{a}ln\left|\sqrt{1+a^2x^2}\ +ax\right|\right) = \frac{1}{a}\ .\frac{\frac{1}{2}(1+a^2x^2)^{-\frac{1}{2}}(2a^2x)+a}{\sqrt{1+a^2x^2}\ +ax}$

Writing $A = \sqrt{1+a^2x^2}$, this gives:

$\frac{A^{-1}ax+1}{A+ax} = \frac{1}{A}\ .\frac{ax+A}{A+ax} = \frac{1}{A}$, as required