# Determinants & Inverses (13 pages; 4/9/18)

## Contents



## (A) Determinants

### (1) Miscellaneous

(i) $|AB| = |A||B|$ (see (ix) for proof)

(ii) $|M||M^{-1}| = |MM^{-1}| = |I| = 1$, so that $|M^{-1}| = \frac{1}{|M|}$

(iii) $|M^T| = |M|$

(iv) Adding a multiple of one column/row to another column/row leaves the determinant unchanged.

For a $3\times3$ matrix $M$, with columns $\underline{a}$ , $\underline{b}$ & $\underline{c}$,  $|M| = \underline{a}.(\underline{b}\times\underline{c})$

and $|\underline{a} + k\,\underline{b}\,, \underline{b}\,, \underline{c}| = (\underline{a} + k\underline{b}).(\underline{b}\times\underline{c})$

$= \underline{a}.(\underline{b}\times\underline{c}) + (k\underline{b}).(\underline{b}\times\underline{c}) = \underline{a}.(\underline{b}\times\underline{c})$ ,

as $\underline{b}\times\underline{c}$ is perpendicular to $\underline{b}$

(v) Swapping 2 columns/rows changes the sign

(vi) If 2 rows/columns are identical, then the determinant is zero [follows from (v)]

(vii) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix} = \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}$

ie the determinant is unchanged if the columns (or rows) are interchanged cyclicly [follows from (v)]

(viii) $|k\underline{a}\ , \underline{b}\ , \underline{c}| = k|\underline{a}\ , \underline{b}\ , \underline{c}|$

as $(k\underline{a}).(\underline{b} \times \underline{c}) = k[\underline{a}.(\underline{b} \times \underline{c})]$

(ix) The determinant of a $3 \times 3$ matrix is the volume scale factor of the associated transformation. This provides a proof for

$|AB| = |A||B|$ , when successive transformations are applied.

## (2) Manipulating determinants

[The entries of the determinant may be numbers or letters.]

### (i) Gaussian elimination

[This is a standard procedure, which always works, but usually involves awkward fractions. It is also used (using matrices, rather than determinants) to solve simultaneous equations. Generally though, it is simpler to work out the determinant by the usual expansion.]

The aim is to obtain a triangular form, such as $\begin{vmatrix} 2 & 4 & 7 \\ 0 & 2 & 5 \\ 0 & 0 & 5 \end{vmatrix}$

[When solving simultaneous equations which have been reduced to a triangular form, such as

$\begin{pmatrix} 2 & 4 & 7 \\ 0 & 2 & 5 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 8 \end{pmatrix}$, $z$ is obtained from the 3rd row;

then $y$ from the 2nd, and $x$ from the 1st.]

**Example**: $\begin{vmatrix} 2 & 4 & 7 \\ 3 & 5 & 8 \\ 4 & 7 & 9 \end{vmatrix}$

Step 1: $R2 \rightarrow R2 - \frac{3}{2}R1$, to give $\begin{vmatrix} 2 & 4 & 7 \\ 0 & -1 & -\frac{5}{2} \\ 4 & 7 & 9 \end{vmatrix}$

or $-\frac{1}{2}\begin{vmatrix} 2 & 4 & 7 \\ 0 & 2 & 5 \\ 4 & 7 & 9 \end{vmatrix}$

Step 2: $R3 \rightarrow R3 - \frac{4}{2}R1$, to give $-\frac{1}{2}\begin{vmatrix} 2 & 4 & 7 \\ 0 & 2 & 5 \\ 0 & -1 & -5 \end{vmatrix}$

Step 3: $R3 \rightarrow R3 - \frac{(-1)}{2}R2$, to give $-\frac{1}{2}\begin{vmatrix} 2 & 4 & 7 \\ 0 & 2 & 5 \\ 0 & 0 & -\frac{5}{2} \end{vmatrix}$

or $\frac{1}{4}\begin{vmatrix} 2 & 4 & 7 \\ 0 & 2 & 5 \\ 0 & 0 & 5 \end{vmatrix} = \frac{1}{4}(2)(2)(5) = 5$

**(ii) Alternative strategy**

In many cases it is possible to take advantage of some of the following devices. These are illustrated in the examples below.

(a) Take out any common factors of a row or column.

(b) If an entry is repeated in a row or column, subtract the appropriate column or row, to produce a zero entry.

(c) Once a row such as $(1, 0, 0)$ has been obtained, multiples of this can be subtracted from other rows to produce more zeros.

## (iii) Examples

### Example 1

Factorise $\Delta= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

### Solution

Replacing R1 with R1 - R2 (where R1 means row 1),

$$\Delta= \begin{vmatrix} 0 & x-y & x^2-y^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

This has the advantage of creating a 0 in R1 and obtaining a common factor for the other elements of the row.

Thus $\Delta= (x-y) \begin{vmatrix} 0 & 1 & x+y \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

Similarly, we can replace R2 with R2 - R3, to give

$$(x-y) \begin{vmatrix} 0 & 1 & x+y \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$$

so that $\Delta= (x-y)(y-z) \begin{vmatrix} 0 & 1 & x+y \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix}$

$= (x-y)(y-z)(y+z-x-y) = (x-y)(y-z)(z-x)$

**Example 2**

Show that $\begin{vmatrix} -2 & 1 & 2k \\ -1 & 1 & k+1 \\ 2 & k-1 & 1 \end{vmatrix}$ is independent of $k$

**Solution**

(The strategy used here is to create zeros wherever possible.)

$$R1 \to R1 - R2 \Rightarrow \begin{vmatrix} -1 & 0 & k-1 \\ -1 & 1 & k+1 \\ 2 & k-1 & 1 \end{vmatrix}$$

$$R2 \to R2 - R1 \Rightarrow \begin{vmatrix} -1 & 0 & k-1 \\ 0 & 1 & 2 \\ 2 & k-1 & 1 \end{vmatrix}$$

$$R3 \to R3 + 2 \times R1 \Rightarrow \begin{vmatrix} -1 & 0 & k-1 \\ 0 & 1 & 2 \\ 0 & k-1 & 2k-1 \end{vmatrix}$$

$$= -1(2k - 1 - 2k + 2) = -1$$

**Example 3**

Evaluate $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ by row and column operations

**Solution**

**Method 1**

$$R2 \to R2 - 4 \times R1 \Rightarrow \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{vmatrix}$$

[Note: Had the 1 not been present, we could have created it by taking out the appropriate factor from the 1st row or column.]

$$R3 \to R3 - 7 \times R1 \Rightarrow \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix}$$

$$R3 \to R3 - 2 \times R2 \Rightarrow \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{vmatrix}$$

$= 0$

**Method 2**

$$C2 \to C2 - C1 \Rightarrow \begin{vmatrix} 1 & 1 & 3 \\ 4 & 1 & 6 \\ 7 & 1 & 9 \end{vmatrix}$$

$C3 \to C3 - C1 \Rightarrow \begin{vmatrix} 1 & 1 & 2 \\ 4 & 1 & 2 \\ 7 & 1 & 2 \end{vmatrix} = 0$ , as two columns are equal

## (iv) Use of the Factor theorem

To factorise $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$:

$\Delta$ can be considered as a polynomial in $x$ (for example); $f(x)$ say.

As $f(y) = \begin{vmatrix} 1 & y & y^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$ (as rows 1 & 2 are equal), it follows

that $(x - y)$ is a factor of $\Delta$.

Similarly $(x - z)$ is a factor, and also $(y - z)$ (by symmetry: considering $\Delta$ as a polynomial in $y$ or $z$).

We can write $\Delta = a(x-y)(x-z)(y-z)$. There won't be any other factors involving $x, y\ or\ z$, since if we expand $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ by the 1st column, all the terms are seen to be cubic in $x, y$ & $z$.

Then, as one of the terms of the expansion is $x^2(-y)$ [expanding by the 1st row, for example], $a$ must be $-1$,

and $\Delta = -(x-y)(x-z)(y-z)$, or $\Delta = (x-y)(y-z)(z-x)$

## (v) Alternative method (when the elements are numbers)

The matrix is extended, by adding on repeats of the 2nd and 3rd rows, as shown below. For each upward or downward diagonal

with 3 elements, the elements are multiplied together, and the sign of the resulting product is reversed in the case of the downward diagonals.

**Example**: To find $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

1 2 3
4 5 6
7 8 9

1 2 3
4 5 6

$\Delta = (7)(5)(3) + (1)(8)(6) + (4)(2)(9)$

$-(1)(5)(9) - (4)(8)(3) - (7)(2)(6)$

$= 105 + 48 + 72 - 45 - 96 - 84 = 0$

## (3) Cramer's rule

This provides a compact way of solving simultaneous equations. For two equations, expressed in matrix form as

$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$, the solution is:

$$x = \frac{\begin{vmatrix} e & c \\ f & d \end{vmatrix}}{\begin{vmatrix} a & c \\ b & d \end{vmatrix}} \ \& \ y = \frac{\begin{vmatrix} a & e \\ b & f \end{vmatrix}}{\begin{vmatrix} a & c \\ b & d \end{vmatrix}}$$

and the same method applies to larger numbers of equations.

## (B) Transposes

[denoted by $A^T$ *or* $A'$]

(i) $(AB)^T = B^T A^T$

Note that $A$ & $B$ don't need to be square matrices (provided that $AB$ is defined).

(ii) $AA^T$ is symmetric

as $(AA^T)^T = (A^T)^T A^T = AA^T$

(iii) $(A^T)^{-1} = (A^{-1})^T$ (see Matrices - Exercises (Part 1))

(iv) $|A^T| = |A|$

(v) $A$ is described as orthogonal if $A^{-1} = A^T$

(vi) Scalar products

If $\underline{x}_1$ & $\underline{x}_2$ are two column vectors, their scalar product is given by

${\underline{x}_1}^T \underline{x}_2$ (or ${\underline{x}_2}^T \underline{x}_1$ ; the effect of taking the transpose of ${\underline{x}_1}^T \underline{x}_2$, which (as it is a scalar) leaves it unchanged); ie a row vector multiplied (on its right) by a column vector. [This gives a

$(1 \times n) \times (n \times 1) = 1 \times 1$ matrix; ie a scalar. The reverse order would be incorrect, giving a

$(n \times 1) \times (1 \times n) = n \times n$ matrix.]

**(C) $3 \times 3$ Inverses**

Let $|\text{A}| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 \\ 5 & 1 & 4 \\ 6 & 2 & 3 \end{vmatrix}$, for example

$= 1(1 \times 3 - 2 \times 4) - 5(3 \times 3 - 2 \times 2) + 6(3 \times 4 - 1 \times 2)$ (1)

(expanding by the 1st column).

Let $A_1 = (1 \times 3 - 2 \times 4)$, $A_2 = -(3 \times 3 - 2 \times 2)$ ,

$A_3 = (3 \times 4 - 1 \times 2)$

or $A_1 = +\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ , $A_2 = -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$, $A_3 = +\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$ in general.

These are referred to as the cofactors of $a_1, a_2, a_3$, respectively.

Other cofactors are defined similarly, with the + and − signs alternating as we go round the matrix. Thus:

$B_1 = -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$, $B_2 = +\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$, $B_3 = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

$$C_1 = +\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, C_2 = -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}, C_3 = +\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

The cofactors are thus **signed** $2 \times 2$ determinants. The **un**signed $2 \times 2$ determinants are referred to as the **minors** of $a_1, a_2, a_3$ respectively.

So, referring back to (1), $a_1A_1 + a_2A_2 + a_3A_3 = |A|$

Similarly it can be shown that

$b_1B_1 + b_2B_2 + b_3B_3 = |A|$ and $c_1C_1 + c_2C_2 + c_3C_3 = |A|$

Also $b_1A_1 + b_2A_2 + b_3A_3 =$

$3(1 \times 3 - 2 \times 4) - 1(3 \times 3 - 2 \times 2) + 2(3 \times 4 - 1 \times 2)$

$= \begin{vmatrix} 3 & 3 & 2 \\ 1 & 1 & 4 \\ 2 & 2 & 3 \end{vmatrix}$ (expanding by the 1st column) $= 0$

(as two of the columns of the determinant are identical).

Thus $b_1A_1 + b_2A_2 + b_3A_3 = 0$ (property of 'alien cofactors'), and similarly for other combinations of letters, so that

eg $c_1B_1 + c_2B_2 + c_3B_3 = 0$

Now consider the matrix of cofactors: $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$

and form the transpose: $\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$

This transpose is called the **adjugate** or **adjoint** of the original matrix.

Then, from the above, it follows that

$$\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}$$

$$= |A|\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence $A^{-1} = \frac{1}{|A|}\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$

**Summary of Procedure**

(1) Find cofactors of each cell (**signed** $2 \times 2$ determinants).

(2) Create the **transpose** of the matrix of cofactors.

(3) Divide by |A|.

**Exercise 1**: Find $\begin{pmatrix} 1 & 3 & 2 \\ 5 & 1 & 4 \\ 6 & 2 & 3 \end{pmatrix}^{-1}$

## Solution

Step 1: Matrix of cofactors $= \begin{pmatrix} -5 & 9 & 4 \\ -5 & -9 & 16 \\ 10 & 6 & -14 \end{pmatrix}$

Step 2: Transpose of matrix of cofactors $= \begin{pmatrix} -5 & -5 & 10 \\ 9 & -9 & 6 \\ 4 & 16 & -14 \end{pmatrix}$

Step 3: $\begin{pmatrix} 1 & 3 & 2 \\ 5 & 1 & 4 \\ 6 & 2 & 3 \end{pmatrix}^{-1} = \frac{1}{30}\begin{pmatrix} -5 & -5 & 10 \\ 9 & -9 & 6 \\ 4 & 16 & -14 \end{pmatrix}$

[As a check, $\frac{1}{30}\begin{pmatrix} -5 & -5 & 10 \\ 9 & -9 & 6 \\ 4 & 16 & -14 \end{pmatrix}\begin{pmatrix} 1 & 3 & 2 \\ 5 & 1 & 4 \\ 6 & 2 & 3 \end{pmatrix}$ should equal $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$]

## Exercise 2

Show that $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} = \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}$

**Solution**

$a_1A_1 + b_1B_1 + c_1C_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (expanding by 1st row) $= |A|$

$a_1A_2 + b_1B_2 + c_1C_2 = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (expanding by 1st row) $=$ 0,

as two of the rows are identical; and similarly for other cells.