Differential Equations - models (4 pages; 14/9/19)

See also: " Differential Equations - Oscillations"

(1) Example: Population growth

If a population increases at a rate that is proportional to its size, then

$\frac{dP}{dt} = kP \ (k > 0)$

Solution of the differential equation:

P= $P_0 e^{kt}$, where P_0 is the initial population (see "First Order Differential Equations").

(2) Example: Cup of tea cooling

According to Newton's law of cooling, the rate at which the tea cools is proportional to the difference between its temperature and that of its surroundings (the ' ambient temperature ').

$\frac{dT}{dt} = -k(T - T_A)$, where T_A is the ambient temperature

Solution of the differential equation:

$T = T_A + (T_0 - T_A)e^{-kt}$, where T_0 is the initial temperature of the tea.

(3) As well as parameters such as k and T_A (appearing in the differential equation above), the **general solution** of a differential equation involves one or more constants of integration - one for a first order equation, two for a second order equation etc (the **order** of a differential equation is the order of its highest

derivative). The general solution can be represented by a family of curves.

Initial or **boundary conditions** are used to eliminate the constants of integration, and obtain a **particular solution** (which can be represented by one member of the family of curves).

Solutions of a differential equation can be verified by substitution in the original equation.

(4) Example: Mixing - Type A [my terminology]

A tank initially contains 100 litres of water. Dyed water is added to the tank at a rate of 5 litres per hour. One litre of the dyed water contains 4mg of dye. (The dye can be assumed to disperse through the tank straightaway, and have a negligible effect on the volume.) Liquid is removed from the tank at a rate of 3 litres per hour.

Let x be the number of mg of dye in the tank at time t. Create a differential equation involving x and t.

Solution

If the removal of liquid is ignored for the moment, then

$\frac{dx}{dt} = 4 \times 5 = 20$ mg per hour.

By time t, the amount of liquid in the tank will be

$100 + 5t - 3t = 100 + 2t$ litres.

The amount of dye removed per hour is therefore $x(\frac{3}{100+2t})$ mg.

Hence the required differential equation is

$\frac{dx}{dt} = 20 - \frac{3x}{100+2t}$

(5) Example: Mixing - Type B

A tank initially contains 100 litres of helium. A mixture of oxygen and helium is added to the tank at a rate of 5 litres per hour. The mixture is made up of 60% oxygen and 40% helium. The two gases can be assumed to disperse through the tank straightaway. Gas is removed from the tank at a rate of 3 litres per hour.

Let x be the number of litres of oxygen in the tank at time t. Create a differential equation involving x and t.

Solution

By time t, the total amount of gas in the tank will be

$100 + 5t - 3t = 100 + 2t$,

so that $\frac{dx}{dt} = 0.60 \times 5 - x(\frac{3}{100+2t}) = 3 - \frac{3x}{100+2t}$

[Note: For this example, we can write either

(a) $x(\frac{3}{100+2t})$: if all of the gas is being removed every hour, then x litres of oxygen are being removed; but, as only 3 litres out of $100 + 2t$ are in fact being removed, we need to scale the x down by the factor $\frac{3}{100+2t}$

(b) $3(\frac{x}{100+2t})$: 3 litres of gas are being removed, but only x out of a total of $100 + 2t$ is oxygen, so we need to scale the 3 down by the factor $\frac{x}{100+2t}$

However, for the Type A problem, we couldn't write $3(\frac{x}{100+2t})$, as x and $100 + 2t$ are not the same type of measure: x is a number of mg, whilst $100 + 2t$ is a number of litres.]

Thus, the same form of equation is obtained for both types of mixing (though x is defined differently for the two types).

[In the Pearson Edexcel book "Further Maths: Core Pure 2" (1st Edition), Exercise 8A, Q7 (on p175), I believe the 2nd sentence ought to read "The gas leaks out ...", instead of "The helium leaks out ...".]

(6) Predator-prey model

This commonly take the form of a pair of linear equations, such as $\frac{dx}{dt} = ax + by$, $\frac{dy}{dt} = cy - ex$, with a, b, c & $e > 0$

(x is the population of the predator; its growth rate increases with the size of its own population (due to breeding), and with the size of the prey population (which provides food); the growth rate of the prey population y increases with the size of its own population (due to breeding), and reduces as the predator population increases.)

See "Linear Systems of DEs" for the solution of pairs of linear equations.