Differential Equations: Approximate methods

(4 pages; 18/11/18)

(1) Tangent Fields

Whether or not an **analytical** (ie non-approximate) solution exists for a differential equation of the form $\frac{dy}{dx} = f(x, y)$, it will be possible to plot the **direction indicators** for the curve.

Example: $\frac{dy}{dx} = x + y$

Figure 1 below shows the direction indicators at various points, whilst Figure 2 shows the family of solutions of the equation. (This can be shown to be $y = Ae^x - x - 1$.)

An **isocline** is a locus of points for which the direction indicators are the same. Here, for example, the line $y = -x$ is an isoscline where the gradient of the direction indicator is 0.

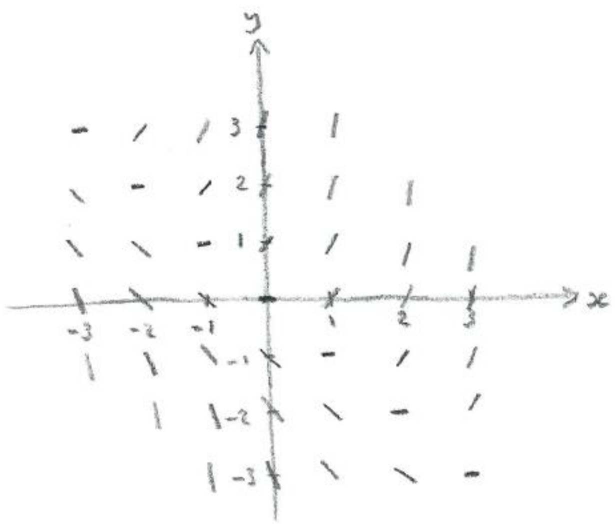

Figure 1

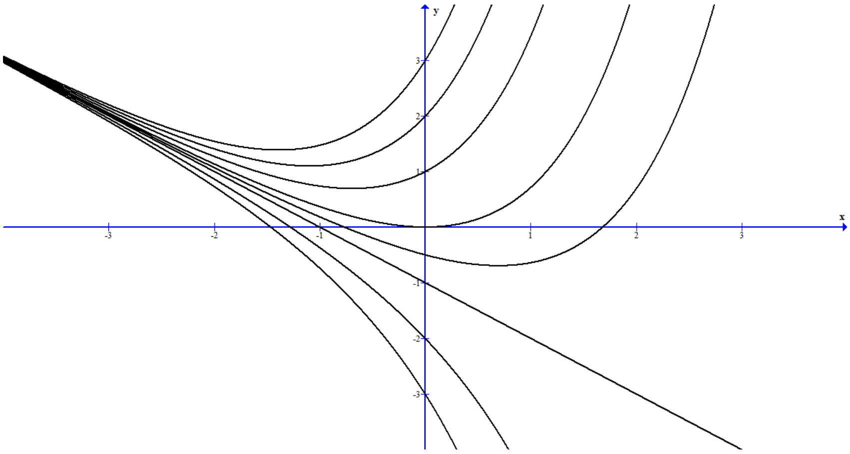

Figure 2

(2) Euler's Method

Referring to Figure 1 below, where AB is the tangent to the curve at A, B is an approximation to B' and is an approximation to C'.

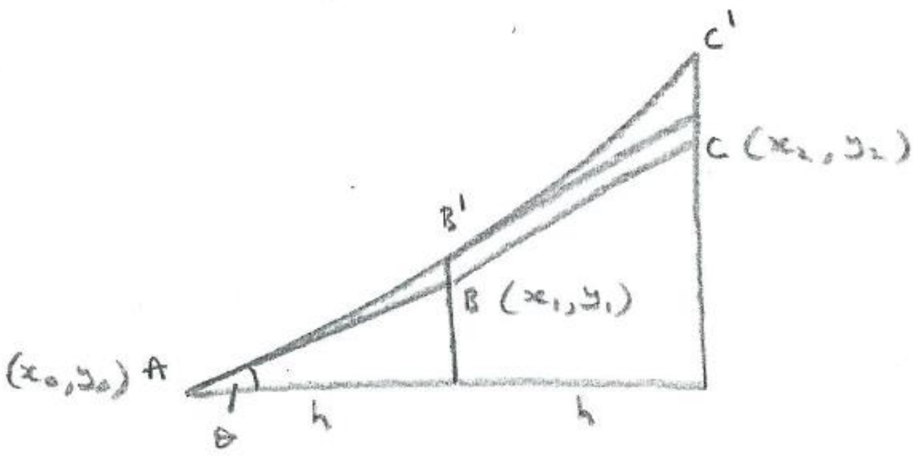

Figure 1

$x_1 = x_0 + h$ and $y_1 = y_0 + h\tan\theta = y_0 + hf(x_0, y_0)$

where $f(x, y) = \frac{dy}{dx}$

Similarly $x_2 = x_1 + h$ and $y_2 = y_1 + hf(x_1, y_1)$ etc

Example: $\frac{dy}{dx} = x + y$

With $x_0 = 0$, $y_0 = 0$ and $h = 0.1$,

$x_1 = 0.1$, $y_1 = 0 + 0.1(0 + 0) = 0$

$x_2 = 0.2$, $y_2 = 0 + 0.1(0.1 + 0) = 0.01$

$x_3 = 0.3$, $y_3 = 0.01 + 0.1(0.2 + 0.01) = 0.031$

If α is a particular value of x , then, for small values of h, it can be shown that the estimate of y at $x = \alpha$, $y(\alpha)$ is approximately a linear function of h; i.e. $y(\alpha) = ah + b$ (*)

By carrying out Euler's method for two values of h, and obtaining a value for $y(\alpha)$ in each case, two simultaneous equations of the form (*) are created, and these can be solved to obtain a value for b. This value is then equivalent to putting $h = 0$, and is thus an improved value for $y(\alpha)$.

For the example above, with $h = 0.05$,

$x_1 = 0.05$, $y_1 = 0 + 0.05(0 + 0) = 0$

$x_2 = 0.1$, $y_2 = 0 + 0.05(0.05 + 0) = 0.0025$

$x_3 = 0.15$, $y_3 = 0.0025 + 0.05(0.1 + 0.0025) = 0.007625$

$x_4 = 0.2$, $y_4 = 0.007625 + 0.05(0.15 + 0.007625) = 0.01550625$

Thus, with $\alpha = 0.2$, $0.01550625 = a(0.05) + b$ (1)

and, for $h = 0.1$(obtained earlier), $0.01 = a(0.1) + b$ (2)

$2 \times (1) - (2)$ then gives $0.0310125 - 0.01 = b$, and hence an improved estimate for $y(0.2)$ is 0.0210 (3sf)

The true value is $e^{0.2} - 0.2 - 1 = 0.0214$ (3sf)