Differential Equations: Approximate methods

(10 pages; 21/1/20)

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(1) Tangent Fields

Whether or not an **analytical** (ie non-approximate) solution exists for a differential equation of the form $\frac{dy}{dx} = f(x, y)$, it will be possible to plot the **direction indicators** for the curve.

Example 1: $\frac{dy}{dx} = x + y$

Figure 1 below shows the direction indicators at various points, whilst Figure 2 shows the family of solutions of the equation. (This can be shown to be $y = Ae^x - x - 1$.)

An **isocline** is a locus of points for which the direction indicators are the same. Here, for example, the line $y = -x$ is an isoscline where the gradient of the direction indicator is 0.

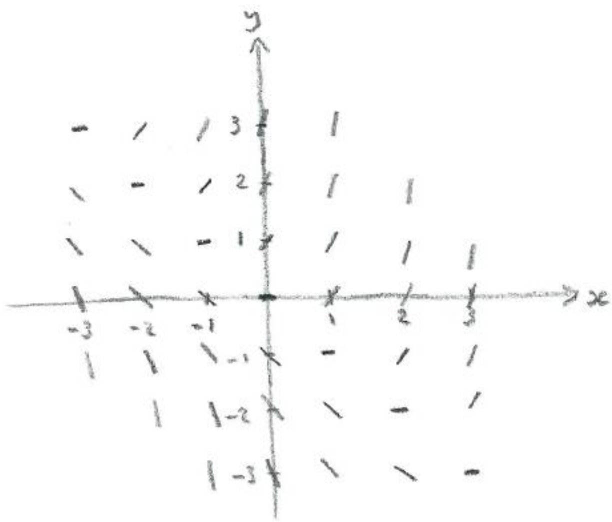


Figure 1

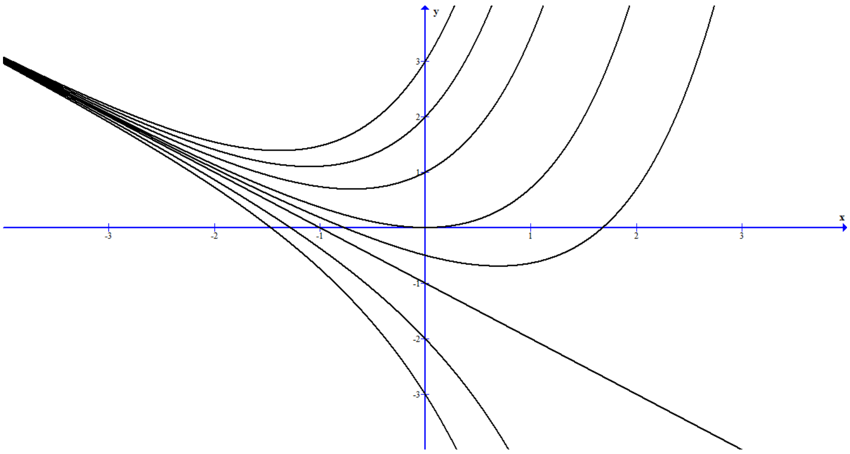
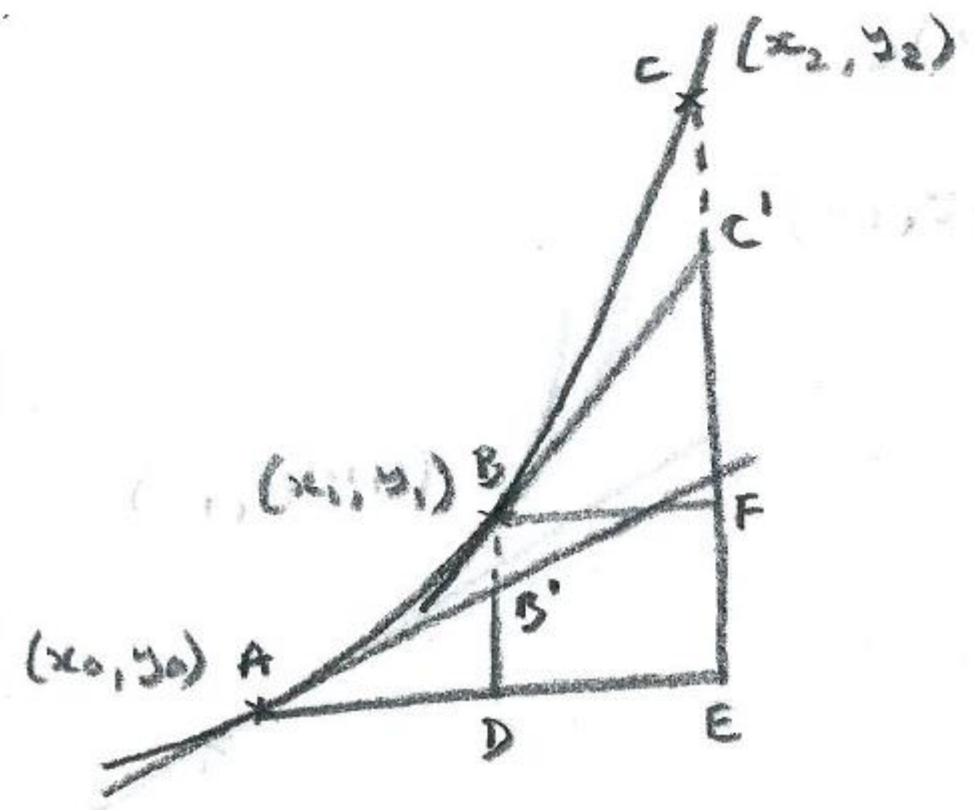


Figure 2

(2) **Euler's Method** (for 1st order equations)

Referring to the diagram below, AB' is the tangent to the curve at A, and $x_1 = x_0 + h$.



$\frac{dy}{dx}|_{x=x_0} = \frac{B'D}{AD} \approx \frac{BD}{AD} = \frac{y_1-y_0}{x_1-x_0} = \frac{y_1-y_0}{h}$ or $y_1 \approx y_0 + h\frac{dy}{dx}|_0$

Also, BC' is the tangent to the curve at B, and $x_2 = x_1 + h$.

And $\frac{dy}{dx}|_{x=x_1} = \frac{C'F}{BF} \approx \frac{CF}{BF} = \frac{y_2-y_1}{x_2-x_1} = \frac{y_2-y_1}{h}$ or $y_2 \approx y_1 + h\frac{dy}{dx}|_1$

and so on.

(Note however, that the value of y_1 used in the estimate of y_2 will itself be an estimate, and that the value of $\frac{dy}{dx}|_1$ will be obtained from the differential equation, and will be based on the estimated value of y_1. The positions of B' and C will therefore be distorted.)

Given a differential equation of the form $\frac{dy}{dx} = f(x, y)$, and a value for y_0, the iterative formula

$$\boxed{\boldsymbol{y_r \approx y_{r-1} + h\frac{dy}{dx}|_{r-1}}}$$

can be applied to obtain approximate values of y_r for $r \geq 1$.

Note: The method will be more accurate when the gradient is not changing that rapidly; ie where the magnitude of $\frac{d^2y}{dx^2}$ is small. In other words, the accuracy depends on the degree of concavity (where $\frac{d^2y}{dx^2} < 0$) or convexity (where $\frac{d^2y}{dx^2} > 0$).

Example 2: $\frac{dy}{dx} = x + y$, where $y = 0$ when $x = 0$

With $x_0 = 0$, $y_0 = 0$ and $h = 0.1$,

$x_1 = 0.1$, $y_1 = 0 + 0.1(0 + 0) = 0$

$x_2 = 0.2$, $y_2 = 0 + 0.1(0.1 + 0) = 0.01$

$x_3 = 0.3$, $y_3 = 0.01 + 0.1(0.2 + 0.01) = 0.031$

Notes

(i) Considering the triangle ADB' in the diagram, $\frac{dy}{dx}|_0$ can be thought of as the scaling factor that has to be applied to AD, in order to obtain $B'D$.

(ii) Given that the true value of y_0 is known, $\frac{dy}{dx}|_0$ obtained from $\frac{dy}{dx} = x + y$ will be the true gradient of the curve at (x_0, y_0); but $\frac{dy}{dx}|_1$ will only be an approximation to the gradient at (x_1, y_1), as it is based on the approximate value y_1.

(3) Improved estimate for Euler's method

If α is a particular value of x , then, for small values of h, it can be shown that the estimate of y at $x = \alpha$, $y(\alpha)$ is approximately a linear function of h; i.e. $y(\alpha) \approx mh + c$ (*)

By carrying out Euler's method for two values of h, and obtaining a value for $y(\alpha)$ in each case, two simultaneous equations of the form (*) are created, and these can be solved to obtain a value for c. This value is then equivalent to putting $h = 0$, and is thus an improved value for $y(\alpha)$.

For Example 2, with $h = 0.05$,

$x_1 = 0.05,\ \ y_1 = 0 + 0.05(0 + 0) = 0$

$x_2 = 0.1,\ \ y_2 = 0 + 0.05(0.05 + 0) = 0.0025$

$x_3 = 0.15,\ \ y_3 = 0.0025 + 0.05(0.1 + 0.0025) = 0.007625$

$x_4 = 0.2,\ \ y_4 = 0.007625 + 0.05(0.15 + 0.007625) = 0.01550625$

Thus, with $\alpha = 0.2$ and $h = 0.05$, an estimate for $y(0.2)$ is 0.01550625 or 0.0155 (3sf)

We can then write $0.01550625 = m(0.05) + c$ (1)

Earlier we obtained the estimate of 0.01 for $y(0.2)$, with $h = 0.1$,

and this gives $0.01 = m(0.1) + c$ (2)

Then $2 \times (1) - (2)$ gives $c = 0.0310125 - 0.01 = 0.0210125$, and hence an improved estimate for $y(0.2)$ is 0.0210 (3sf)

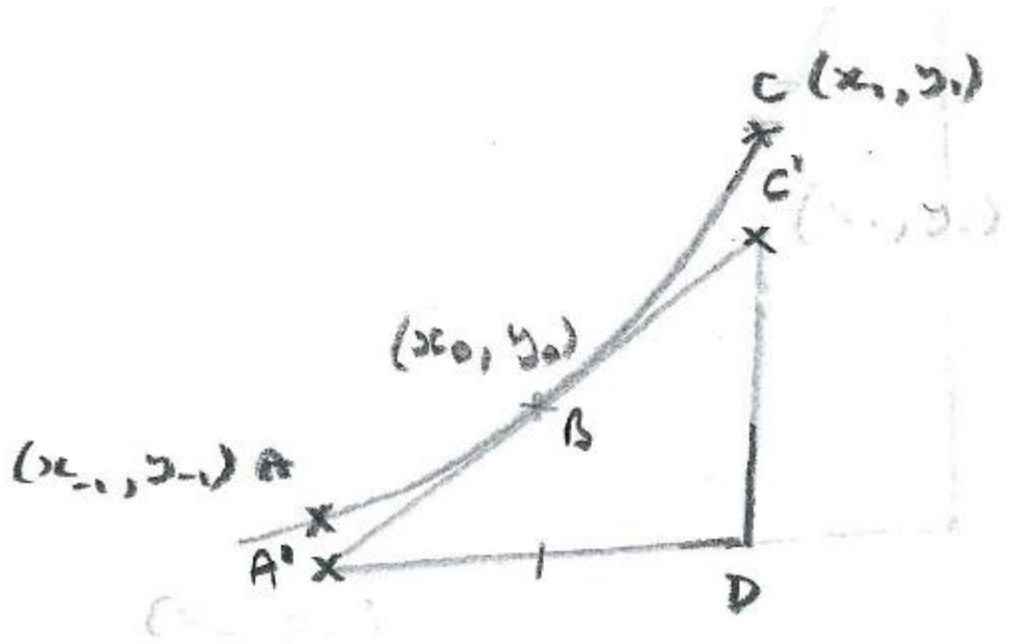
The true value is $e^{0.2} - 0.2 - 1 = 0.0214$ (3sf).

To summarise:

	estimate of $y(0.2)$
$h = 0.1$	0.01
$h = 0.05$	0.0155
$h \approx 0$	0.0210
	0.0214 (true value)

(4) Midpoint method (for 1st order equations)

An improvement can usually be made to Euler's method by considering approximations to the y-coordinate either side of y_0 (see diagram below, where $A'BC'$ is the tangent to the curve at B).



Then $\frac{dy}{dx}|_0 = \frac{C'D}{A'D} \approx \frac{y_1 - y_{-1}}{x_1 - x_{-1}} = \frac{y_1 - y_{-1}}{2h}$ or $y_1 \approx y_{-1} + 2h\frac{dy}{dx}|_0$

This produces the iterative formula:

$$\boxed{\boldsymbol{y_r \approx y_{r-2} + 2h\frac{dy}{dx}|_{r-1}}}$$

The midpoint method requires two initial values of y in order to carry out the iterations (y_1 will be needed in order to obtain an approximate value for $\frac{dy}{dx}|_1$, so that the above formula can be used to find y_2). If only one value is provided, then Euler's method may be used to find an approximate value for the second.

Example 2 (again): $\frac{dy}{dx} = x + y$, where $y = 0$ when $x = 0$

with $h = 0.1$ again.

Solution

$x_0 = 0$, $y_0 = 0$ again.

$x_1 = 0 + 0.1 = 0.1$

Euler's method is applied to find y_1, to give $y_1 = 0$, as before.

Then $\frac{dy}{dx}|_1 \approx x_1 + y_1 = 0.1 + 0 = 0.1$

$x_2 = 0.1 + 0.1 = 0.2$

By the midpoint formula, $y_2 \approx y_0 + 2h\frac{dy}{dx}|_1$,

so that $y_2 \approx 0 + 2(0.1)(0.1) = 0.02$

$x_3 = 0.2 + 0.1 = 0.3$

Then $\frac{dy}{dx}|_2 \approx x_2 + y_2 = 0.2 + 0.02 = 0.22$

$y_3 \approx y_1 + 2h\frac{dy}{dx}|_2$,

so that $y_3 \approx 0 + 2(0.1)(0.22) = 0.044$

[This compares with 0.031 by Euler's method.]

(5) 2nd Order method

Iterative formula:

$$\boxed{\boldsymbol{y_r = -y_{r-2} + 2y_{r-1} + h^2 \frac{d^2y}{dx^2}\Big|_{r-1}}}$$

Derivation: As $\frac{d^2y}{dx^2}$ is the gradient of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}\Big|_0 \approx \frac{\frac{dy}{dx}\big|_0 - \frac{dy}{dx}\big|_{-1}}{h}$

[Notice that we are looking backwards this time, whereas Euler's method looks forward with $\frac{dy}{dx}\Big|_0 = \frac{y_1 - y_0}{h}$]

$$= \frac{\left(\frac{y_1-y_0}{h}\right)-\left(\frac{y_0-y_{-1}}{h}\right)}{h} = \frac{(y_1-y_0)-(y_0-y_{-1})}{h^2} = \frac{y_1-2y_0+y_{-1}}{h^2}$$

$$\Rightarrow h^2\frac{d^2y}{dx^2}\Big|_0 \approx y_1 - 2y_0 + y_{-1}$$

$$\Rightarrow y_1 \approx -y_{-1} + 2y_0 + h^2\frac{d^2y}{dx^2}\Big|_0 ,$$

which gives the required iterative formula.

Example 3: $\frac{d^2y}{dx^2} = x(x+y)$, given that when $x = 1, y = 2$ and $\frac{dy}{dx} = 1$; with $h = 0.1$

Solution

Method 1 (using Euler's formula)

$x_0 = 1\ ,\ y_0 = 2$

$x_1 = x_0 + h = 1 + 0.1 = 1.1$

Use Euler's method obtain an approximate value for y_1:

$y_1 \approx y_0 + h\frac{dy}{dx}|_0 = 2 + (0.1)(1) = 2.2$

$x_2 = x_1 + h = 1.1 + 0.1 = 1.2$

Then the formula $y_r \approx -y_{r-2} + 2y_{r-1} + h^2\frac{d^2y}{dx^2}|_{r-1}$, together with $\frac{d^2y}{dx^2} = x(x+y)$ gives:

$y_2 \approx -y_0 + 2y_1 + h^2\frac{d^2y}{dx^2}|_1$

$= -y_0 + 2y_1 + h^2x_1(x_1 + y_1)$

$= -2 + 2(2.2) + (0.01)(1.1)(1.1 + 2.2)$

$= 2.4363$

and values for y_3 etc are obtained in the same way.

Method 2 (a more accurate - but longer - approach, using the midpoint formula)

$x_0 = 1$, $y_0 = 2$

$x_1 = x_0 + h = 1 + 0.1 = 1.1$

The 2nd order formula $y_r \approx -y_{r-2} + 2y_{r-1} + h^2\frac{d^2y}{dx^2}|_{r-1}$ gives

$y_1 \approx -y_{-1} + 2y_0 + h^2\frac{d^2y}{dx^2}|_0 = -y_{-1} + 2y_0 + h^2x_0(x_0 + y_0)$

so that $y_1 \approx -y_{-1} + 2(2) + (0.01)(1)(1+2) = -y_{-1} + 4.03$ (1)

whilst the midpoint formula $y_r \approx y_{r-2} + 2h\frac{dy}{dx}|_{r-1}$ gives

$y_1 \approx y_{-1} + 2h\frac{dy}{dx}|_0$

so that $y_1 \approx y_{-1} + 2(0.1)(1) = y_{-1} + 0.2$ (2)

Adding (1) & (2): $2y_1 = 4.23$ and $y_1 = 2.115$

[compared with 2.2 by Method 1]

Then $y_2 \approx -y_0 + 2y_1 + h^2 \frac{d^2y}{dx^2}|_1$

$= -y_0 + 2y_1 + h^2 x_1(x_1 + y_1)$

$= -2 + 2(2.115) + (0.01)(1.1)(1.1 + 2.115)$

$= 2.265365$ [compared with 2.4363 by Method 1]

(6) General Points

(i) Approximate values of y_n shouldn't be given to too many decimal places (though a reasonably large number of dps should be kept in the intermediate calculations).

If accurate values of y_n are required, then the process can be repeated for smaller h, until no further change occurs, to the required number of decimal places.

(ii) Euler's method can be used (eg to find another value of y_n) whenever either (a) a formula is provided for $\frac{dy}{dx}$, or (b) when a value is given for $\frac{dy}{dx}$ for a particular value of x.