

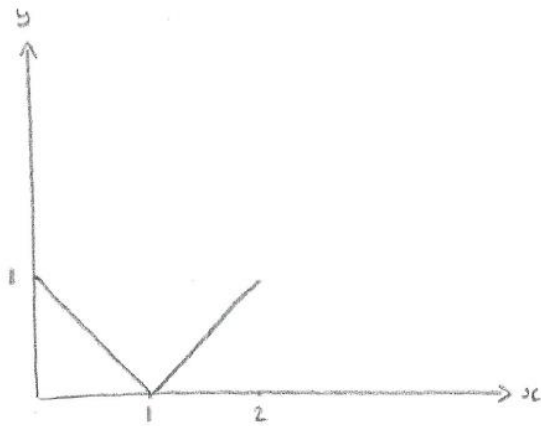
Curve Sketching - Exercises (Sol'ns) (4 pages; 7/10/18)

(1) Sketch the graph of $\sqrt{x^2 - 2x + 1}$ for $0 \leq x \leq 2$

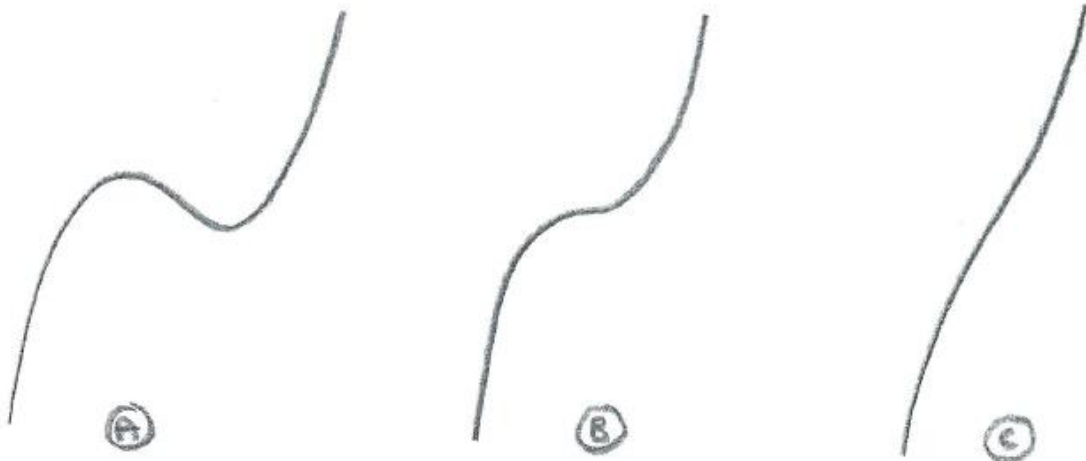
Solution: (see sketch below)

For $0 \leq x \leq 1$, $\sqrt{x^2 - 2x + 1} = \sqrt{(x - 1)^2} = \sqrt{(1 - x)^2} = 1 - x$

For $1 \leq x \leq 2$, $\sqrt{x^2 - 2x + 1} = \sqrt{(x - 1)^2} = x - 1$



(2) (i) What possible shapes might a cubic have (ignoring its position relative to the axes)?



A, B & C have 2, 1 & 0 stationary points respectively. These are for cases where the coefficient of x^3 is positive; so inverted shapes are also possible.

(ii) How many stationary points does the cubic function,

$$f(x) = x^3 + x^2 - 2x + 3 \text{ have?}$$

$$f'(x) = 3x^2 + 2x - 2$$

To find the number of solutions to $f'(x) = 0$,

consider the discriminant: $2^2 - 4(3)(-2) > 0$

Thus there are 2 stationary points.

(iii) What is the condition for there to be 2 stationary points for the general cubic $f(x) = ax^3 + bx^2 + cx + d$?

$$2 \text{ sol'ns of } f'(x) = 3ax^2 + 2bx + c = 0$$

$$\Rightarrow (2b)^2 - 4(3a)c > 0$$

$$\Rightarrow b^2 - 3ac > 0$$

(iv) For $f(x) = ax^3 + bx^2 + cx + d$, find the x -coordinate of any turning points of the gradient.

For a stationary point of the gradient, we want $\frac{d}{dx}(f'(x)) = 0$; ie

$$f''(x) = 0:$$

$$f'(x) = 3ax^2 + 2bx + c$$

$$f''(x) = 6ax + 2b$$

$$f''(x) = 0 \Rightarrow x = -\frac{b}{3a}$$

And $\frac{d^2}{dx^2}(f'(x)) = f'''(x) = 6a > 0$, so that the stationary point is a minimum (ie it is a turning point).

A turning point of the gradient is the definition of a point of inflexion (or inflection).

Thus, all cubics have one point of inflexion. They can be shown to have rotational symmetry about this point.

If the cubic has turning points, how could they be used to find the point of inflexion?

By symmetry, the coordinates of the point of inflexion will be halfway between those of the turning points.

(v) For $f(x) = ax^3 + bx^2 + cx + d$, find conditions for the shape of the curve to be each of the 3 possibilities shown in (i), by considering the gradient at the point of inflexion.

$$f' \left(-\frac{b}{3a} \right) = \frac{b^2}{3a} - \frac{2b^2}{3a} + c = c - \frac{b^2}{3a}$$

Diagram (A): Either (i) $a > 0$ & $f' \left(-\frac{b}{3a} \right) < 0$

or (ii) $a < 0$ & $f' \left(-\frac{b}{3a} \right) > 0$

(i): $3ac - b^2 < 0 \Leftrightarrow b^2 - 3ac > 0$ [agreeing with part (iii)]

(ii): $3ac - b^2 < 0$ also

Diagram (B): Stationary point of inflexion $\Leftrightarrow f' \left(-\frac{b}{3a} \right) = 0$

$$\Leftrightarrow b^2 - 3ac = 0$$

Diagram (C): Either (i) $a > 0$ & $f' \left(-\frac{b}{3a} \right) > 0$

or (ii) $a < 0$ & $f' \left(-\frac{b}{3a} \right) < 0$,

so that $b^2 - 3ac < 0$

(3) Sketch $y = |x - 2| + 1$

Solution

Method 1

Case (i) $x - 2 \geq 0$

$$y = |x - 2| + 1 = (x - 2) + 1 = x - 1 \quad \text{for } x \geq 2$$

Case (ii) $x - 2 < 0$

$$y = |x - 2| + 1 = -(x - 2) + 1 = 3 - x \quad \text{for } x < 2$$

(The two lines will meet when $x = 2, y = 1$)

Method 2

Informally, $y = |x - 2| + 1$ will behave similarly to $y = (x - 2)^2 + 1$, and will have a minimum at $(2, 1)$

