

# Counting Q6 [Problem/H] (9/6/21)

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### Solution

Consider a simpler version of the problem, with just 3 numbers, each of which can be 0, 1 or 2

Then the following are possible:

222, 221, 220, 211, 210, 200

111, 110, 100

000

It may be possible to come up with a recurrence relation of some sort. Let  $f(m, n)$  be the number of possibilities when we can choose between 0, 1, ...,  $m$  for each digit, and there are  $n$  digits in the password.

From the above, we have  $f(2, 3) = f(2, 2) + f(1, 2) + f(0, 2)$

and this can be generalised to

$$f(m, n) = f(m, n - 1) + f(m - 1, n - 1) + \dots + f(0, n - 1)$$

We also note that  $f(m, 1) = m + 1$

Applying this to the problem in question,

$$f(4, 4) = f(4, 3) + f(3, 3) + f(2, 3) + f(1, 3) + f(0, 3)$$

$$= [f(4, 2) + f(3, 2) + \dots + f(0, 2)]$$

$$+ [f(3, 2) + f(2, 2) + \dots + f(0, 2)]$$

$$+ \dots + f(0, 2)$$

$$= f(4, 2) + 2f(3, 2) + 3f(2, 2) + \dots + 5f(0, 2)$$

$$\begin{aligned}
&= [f(4, 1) + f(3, 1) + \dots + f(0, 1)] \\
&+ 2[f(3, 1) + f(2, 1) + \dots + f(0, 1)] \\
&+ \dots 5f(0, 1) \\
&= f(4, 1) + (1 + 2)f(3, 1) + (1 + 2 + 3)f(2, 1) \\
&+ (1 + 2 + 3 + 4)f(1, 1) + (1 + 2 + 3 + 4 + 5)f(0, 1) \\
&= 5 + 3(4) + 6(3) + 10(2) + 15(1) \\
&= 5 + 12 + 18 + 20 + 15 = 70
\end{aligned}$$

[This method can be applied to the original BMO problem, though it would become unwieldy if the length of the password exceeded 6 digits.]

**Alternative (quicker) method** [based on the official solutions, contained in "A Mathematical Olympiad Primer" by Geoff Smith]:

Each possibility can be represented using the following system:

3220 is represented by DXDXXDDX

and 4311 is represented by XDXDDXXD

For 3220, the 1st letter D means that we are dropping by 1 from the maximum of 4, and the 2nd letter X means that we have reached the value of the 1st digit; similarly the next 2 letters DX mean that we are dropping by another 1 to arrive at the 2nd digit; the 5th letter X means that we don't drop at all to arrive at the 3rd digit; the final letters DDX mean that we drop by 2 to arrive at the last digit.

For 4311, we need a D on the end to get down to 0.

In all cases there will be 4 Ds and 4 Xs, and the Ds and Xs can appear in any of the places, so that the number of possibilities is  $\binom{8}{4} = 70$ .

[See the official solutions for a couple of other approaches.]